Lecture 4
Dynamic Model

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Dynamic Model

- Now start our analysis of dynamic general equilibrium models, which we will continue in the rest of the class.
- Today start with optimal allocations, solving social planner problem. Later consider equilibrium analysis.
- In going from static to dynamic model, the main difference is savings and investment.
- Households no longer consume all income each period, save some for future consumption (or borrow against future income).
- Firm no longer have fixed capital stock on hand each period, may choose to invest in order to build up future capital (or disinvest to allow future capital to fall).

Household Preferences

- Representative household lives infinite number of periods.
- Utility function:

$$\begin{aligned} V_0 &= U(c_0) + \beta U(c_1) + \beta^2 U(c_2) + \beta^3 U(c_3) + \dots \\ &= \sum_{t=0}^{\infty} \beta^t U(c_t) \end{aligned}$$

  c_t is consumption at date t
  $\beta \in (0,1)$ **discount factor** measures household's degree of impatience. Define $\beta = \frac{1}{1+\theta}$, where θ is **discount rate**
- Preferences over $\{c_0, c_1, \dots\}$ satisfy the conditions discussed previously, i.e. monotonicity $(U' > 0)$ and convexity $(U'' < 0)$.

More on Preferences

$$V_0 = \sum_{t=0}^{\infty} \beta^t U(c_t)$$

- Abstract from labor/lesiure tradeoff for now. Inelastic labor supply: work full time h hours, yields no utility.
- Consumption smoothing, partially offset by discounting.
- Assume all c_t are normal: more income $\Rightarrow$ more consumption at each date t
- From vantage point of date 0, marginal utility of c_t:

$$MU_{c_t} = \frac{\partial V_0}{\partial c_t} = \beta^t U'(c_t)$$

- Intertemporal marginal rate of substitution measures willingness to substitute consumption over time:

$$MRS_{c_t, c_{t+1}} = \frac{MU_{c_t}}{MU_{c_{t+1}}} = \frac{U'(c_t)}{\beta U'(c_{t+1})}$$

Technology

- Continue to abstract from labor for now. Assume $h = 1$ is supplied inelastically. Then production is:

$$y_t = F(k_t, 1) \equiv F(k_t)$$

where production function is same as before. Note since $N = 1$ fixed, diminishing marginal returns in k:

$$F'(k) > 0, \quad F''(k) < 0$$

- For technical reasons, also assume Inada conditions:

$$\lim_{k \to 0} F'(k) = +\infty, \quad \lim_{k \to \infty} F'(k) = 0$$

Investment

- Firms can now invest in order to expand future productivity.
- Capital depreciates at rate δ, and investment at t increases k_{t+1}:

$$k_{t+1} = (1-\delta)k_t + i_t$$

- We abstract from government spending, so the feasibility or goods market clearing condition now includes investment and consumption:

$$y_t = c_t + i_t$$

- Combining equations gives us the tradeoff between consumption and capital:

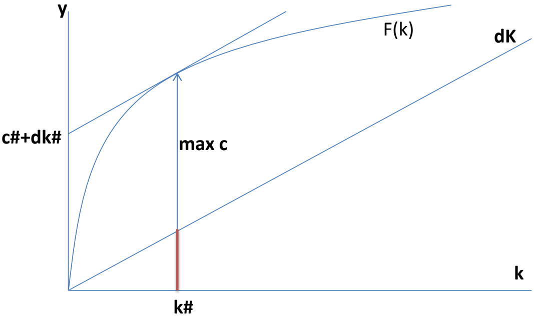
$$c_t = F(k_t) - k_{t+1} + (1-\delta)k_t$$

Steady States

- In general $\{k_t, c_t, y_t, i_t\}$ will vary over time. But let's look for a steady state, where they are constant.
- From the previous expression this implies:

$$c = F(k) - k + (1 - \delta)k = F(k) - \delta k$$

- In the steady state, consumption equals output minus replacement investment δk.



Output, replacement investment, and consumption

The Golden Rule Allocation

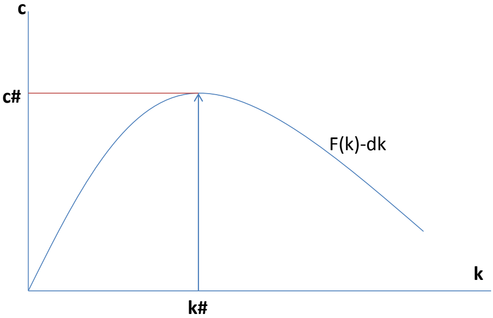
- We now consider the social planner's problem to determine the optimal allocation.
- We first focus on a simple objective, suppose that the planner wanted to maximize utility in the steady state. This is known as the "Golden Rule" allocation, as it treats consumption at all dates equally.

$$\max_{c,k} U(c)$$

$$\text{subject to: } c = F(k) - \delta k$$

- Since $U(c)$ strictly increasing, this is equivalent to max c s.t. $c = F(k) - \delta k$
- First order condition determines golden rule capital $k^{\#}$.

$$F'(k^{\#}) = \delta$$



Net output

Optimal Allocation

- While the golden rule gives the maximal amount of steady state consumption, in general it is not optimal.
- If households are impatient ($\beta < 1$) then they value current consumption more than future consumption. So the timing of consumption matters.
- So now let's consider the optimal allocation:

$$\max_{\{c_t, k_t\}} \sum_{t=0}^{\infty} \beta^t U(c_t)$$

$$\text{subject to: } c_t = F(k_t) - k_{t+1} + (1-\delta)k_t, \quad \forall t, k_0 \text{ given}$$

Characterizing the Optimal Allocation

- Form the Lagrangian with multipliers $\{\lambda_t\}$ on the constraints:

$$\mathcal{L} = \max_{\{c_t, k_t\}} \sum_{t=0}^{\infty} \left(\beta^t U(c_t) + \lambda_t [F(k_t) - k_{t+1} + (1-\delta)k_t - c_t] \right)$$

- First order conditions for any c_t, and for k_{t+1}, $t > 0$:

$$\begin{aligned} \beta^t U'(c_t) &= \lambda_t \\ -\lambda_t + \lambda_{t+1}[F'(k_{t+1}) + 1 - \delta] &= 0. \end{aligned}$$

- Note that if there were a finite terminal date T, we would have $k_{T+1} = 0$. Consume everything in last date.
- For infinite horizon problem need a similar condition known as **transversality** condition:

$$\lim_{T\to\infty} \beta^T U'(c_T) k_{T+1} = 0$$

Value in utility terms of capital goes to zero at infinity.

The Euler Equation

- Combine the two optimality conditions to get:

$$U'(c_t) = \beta U'(c_{t+1})[F'(k_{t+1}) + 1 - \delta]$$

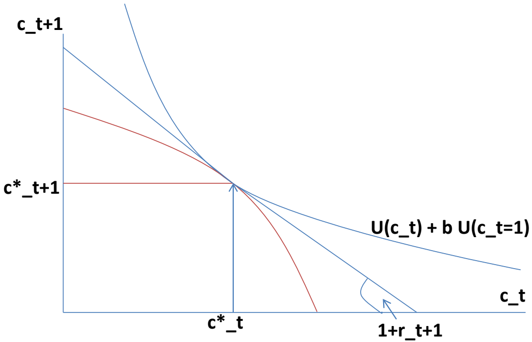
- This is known as an **Euler equation** and is a key condition for optimality in dynamic models.
- Can also be written:

$$MRS_{c_t, c_{t+1}} = \frac{U'(c_t)}{\beta U'(c_{t+1})} = F'(k_{t+1}) + 1 - \delta$$

- Here $F'(k_{t+1}) + 1 - \delta$ is the slope of the production possibility frontier for c_t, c_{t+1}. To see this note:

$$\begin{aligned} c_{t+1} &= F(k_{t+1}) - k_{t+2} + (1-\delta)k_{t+1} \\ &= F(F(k_t) - c_t + (1-\delta)k_t) - k_{t+2} + (1-\delta)[F(k_t) - c_t + (1-\delta)k_t] \end{aligned}$$

- Take derivative with respect to c_t: $F'(k_{t+1}) + 1 - \delta$



Optimal choice

More on The Euler Equation

- Can also interpret $F'(k_{t+1}) - \delta$ as holding period return r_{t+1} on capital.

$$U'(c_t) = \beta U'(c_{t+1})[1 + r_{t+1}] = \beta U'(c_{t+1}) R_{t+1}$$

- Recall that U is concave, so $U'' < 0$ or in other words $U'(c)$ is decreasing.
- So if:

$$\begin{aligned} \beta(1 + r_{t+1}) &> 1, \ U'(c_t) > U'(c_{t+1}), \Rightarrow c_t < c_{t+1} \\ \beta(1 + r_{t+1}) &< 1, \ U'(c_t) < U'(c_{t+1}), \Rightarrow c_t > c_{t+1} \\ \beta(1 + r_{t+1}) &= 1, \ U'(c_t) = U'(c_{t+1}), \Rightarrow c_t = c_{t+1} \end{aligned}$$

- Behavior of consumption over time depends on rate of time preference relative to interest rate.
- If equal, perfect consumption smoothing.

Optimal Steady State

- Look for a steady state of the optimal allocation.

$$U'(c^*) = \beta U'(c^*)[F'(k^*) + 1 - \delta]$$

Or, recalling that $\beta = 1/(1+\theta)$:

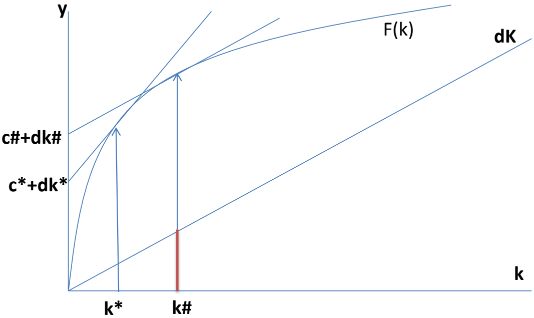
$$F'(k^*) = \frac{1}{\beta} + \delta - 1 = \delta + \theta$$

- From the previous expression we also have:

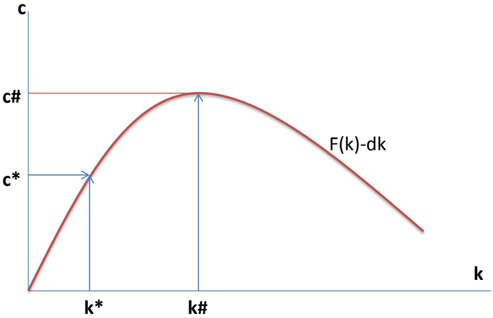
$$c^* = F(k^*) - \delta k^*$$

- The optimal steady state is only equal to the golden rule if $\theta = 0$. And since $F''(k) < 0$ we have:

$$F'(k^{\#}) = \delta < \delta + \theta = F'(k^*), \ \Rightarrow k^{\#} > k^*$$



Optimal steady state consumption and capital



Optimal steady state and golden rule

An Example

- Now work out a parametric example, using standard functional forms. Cobb-Douglas production:

$$y = zk^{\alpha}$$

- For preferences, set:

$$U(c) = \frac{c^{1-\sigma} - 1}{1 - \sigma}$$

For $\sigma > 0$. Interpret $\sigma = 1$ as $U(c) = \log c$.

- These imply the Euler equation:

$$c_t^{-\sigma} = \beta c_{t+1}^{-\sigma}[1 + \alpha z k_{t+1}^{\alpha-1} - \delta] = \beta c_{t+1}^{-\sigma} R_{t+1}$$

- For these preferences σ gives the curvature and so governs how the household trades off consumption over time.

Intertemporal Elasticity of Substitution

- Define intertemporal elasticity of substitution IES as:

$$IES = \frac{d\frac{c_{t+1}}{c_t}}{dR_{t+1}} \frac{R_{t+1}}{\frac{c_{t+1}}{c_t}} = \frac{d\log\left(\frac{c_{t+1}}{c_t}\right)}{d\log R_{t+1}}$$

- Then for these preferences we have:

$$\begin{aligned} c_t^{-\sigma} &= \beta c_{t+1}^{-\sigma} R_{t+1} \\ \Rightarrow \left(\frac{c_{t+1}}{c_t}\right)^{\sigma} &= \beta R_{t+1} \\ \Rightarrow \frac{c_{t+1}}{c_t} &= \beta^{\frac{1}{\sigma}} R_{t+1}^{\frac{1}{\sigma}} \\ \Rightarrow \log\left(\frac{c_{t+1}}{c_t}\right) &= \frac{1}{\sigma}\log\beta + \frac{1}{\sigma}\log(R_{t+1}) \\ \Rightarrow IES &= \frac{1}{\sigma} \end{aligned}$$

Steady State in the Example

- Recall the Euler equation:

$$c_t^{-\sigma} = \beta c_{t+1}^{-\sigma}[1 + \alpha z k_{t+1}^{\alpha-1} - \delta]$$

- Steady state:

$$\begin{aligned} F'(k^*) &= z\alpha(k^*)^{\alpha-1} = \delta + \theta \\ \Rightarrow k^* &= \left(\frac{\alpha z}{\delta + \theta}\right)^{\frac{1}{1-\alpha}} \end{aligned}$$

- Then we get consumption:

$$\begin{aligned} c^* &= z(k^*)^{\alpha} - \delta k^* \\ &= z\left(\frac{\alpha z}{\delta + \theta}\right)^{\frac{\alpha}{1-\alpha}} - \delta\left(\frac{\alpha z}{\delta + \theta}\right)^{\frac{1}{1-\alpha}} \end{aligned}$$