Search Theoretic Models of Money

Economics 714

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Essential Models of Money

• Hahn (1965): money is essential if it allows agents to achieve allocations they cannot achieve with other mechanisms that also respect the enforcement and information constraints in the environment.

• Why do we care about essential models of money?

• Three frictions that will make money essential:
  1. Double-coincidence of wants problem.
  2. Long-run commitment cannot be enforced.
  3. Agents are anonymous: histories are not public information.

• Money is a consequence of these frictions in trade: medium of exchange.
Three Generations of Models

1. 1 unit of money, 1 unit of good: Kiyotaki and Wright (1993).

2. 1 unit of money, endogenous units of good: Trejos and Wright (1995).

3. Endogenous units of money, endogenous units of good: Lagos and Wright (2005).
Environment

- [0, 1] continuum of anonymous agents.
- Live forever and discount future at rate $r$.
- [0, 1] continuum of goods. Good $i$ is produced by agent $i$.
- Goods are non-storable: no commodity money.
- Unit cost of production $c \geq 0$. 
Double-Coincidence of Wants Problem

- I do not produce what I like (non-restrictive: home production, specialization).

- \( iWj \): agent \( i \) likes to consume good produced by agent \( j \):
  1. utility \( u > c \) from consuming \( j \).
  2. utility 0 otherwise.

- Probabilities of matching:

  \[
  p(iWi) = 0 \\
  p(jWi) = x \\
  p(jWi|iWj) = y
  \]
First Generation: Fixed Money and Fixed Good

• Exogenously given quantity $M \in [0, 1]$ of an indivisible unit of storable good.

• Holding money yields zero utility $\gamma$: fiat money.

• Initial endowment: $M$ agents are randomly endowed with one unit of money.

• Agents holding money cannot produce (for example because you need to consume before you can produce again).

• We eliminate (non-trivial) distributions.
Trades

- Pairwise random matching of agents with Poisson arrival time \( \alpha \).

- Bilateral trading is important, randomness is not (Corbae, Temzelides, Wright, 2003).

- Upon meeting, agents decide whether to trade. Then, they part company and re-enter the process.

- History of previous trades is unknown.

- Exchange 1 unit of good for 1 unit of good (barter) or 1 unit of money.
Individual Trading Strategies

• Agents never accept a good in trade if he does not like to consume it since it is not storable.

• They will barter if they like the both agents in the pair like each other goods.

• Would they accept money for goods and viceversa?

• We will look at stationary and symmetric Nash equilibria.
Probabilities

• You meet someone with arrival rate $\alpha$.

• This person can produce with probability $1 - M$.

• With probability $x$ you like what he produces.

• With probability $\pi = \pi_0\pi_1$ (endogenous objects to be determined) both of you want to trade.

• If $\pi > 0$, we say that money circulates.
Value Functions

• Value functions with money, $V_1$:
  
  \[ rV_1 = \alpha x (1 - M) \pi (u + V_0 - V_1) \]

• Value functions without money, $V_0$.
  
  \[ rV_0 = \alpha xy (1 - M)(u - c) + \alpha x M \pi (V_1 - V_0 - c) \]

• Renormalize $\alpha x = 1$ by picking time units:
  
  \[ rV_1 = (1 - M) \pi (u + V_0 - V_1) \]
  \[ rV_0 = y (1 - M)(u - c) + M \pi (V_1 - V_0 - c) \]
Individual Trading Strategies

- Net gain from trading goods for money:
  \[ \Delta_0 = V_1 - V_0 - c = \frac{(1 - M)(\pi - y)(u - c) - rc}{r + \pi} \]

- Net gain from trading money from goods:
  \[ \Delta_1 = u + V_0 - V_1 = \frac{(M\pi + (1 - M)y)(u - c) + ru}{r + \pi} \]
Equilibrium Conditions for $\pi_0$ and $\pi_1$

- Clearly:

$$\pi_j \begin{cases} 
= 1 \\
\in [0, 1] \\
= 0 
\end{cases} \quad \text{as} \quad \Delta_j \begin{cases} 
> 0 \\
= 0 \\
< 0 
\end{cases}$$

- Plug those into the individual trading strategies, and check them.
Characterizing $\pi$

• Clearly $\Delta_1 > 0$. Hence $\pi_1 = 1$, i.e., the agent with money always wants to trade.

• For $\pi_0$, you have

$$\Delta_0 = \frac{(1 - M)(u - c)\pi_0}{r + \pi_0} - \frac{(1 - M)y(u - c) + rc}{r + \pi_0}$$

• Then, $\Delta_0$ has the same sign as

$$\pi_0 - \frac{rc + (1 - M)y(u - c)}{(1 - M)(u - c)} = \pi_0 - \hat{\pi}$$
Multiple Equilibria

• Nonmonetary equilibrium: we have an equilibrium where $\pi_0 = 0$.

• Monetary equilibrium: if

$$c < \frac{(1 - M)(1 - y)}{r + (1 - M)(1 - y)}u$$

then $\hat{\pi} < 1$ and $\pi_0 = 1$ is an equilibrium as well.

• Mixed-monetary equilibrium: $\pi_0 = \pi^\wedge$. However, not robust (Schevchenko and Wright, 2004).
Equilibria in \((\gamma, r)\)-Space When Money Holders Cannot Produce

\(\pi_0 = 0, \pi_1 = 1\)

\(\pi_0 = 0, \pi_0 = 1\) or \(\pi_0 \in (0,1), \pi_1 = 1\)

\(\pi_0 = 1, \pi_1 = 1\)

\(\pi_0 = 1, \pi_1 = 0\) or \(\pi_1 \in (0,1)\)

\(\pi_0 = 1, \pi_1 = 0\)
Welfare

- Define welfare as the average utility:

\[ W = MV_1 + (1 - M)V_0 \]

- Then:

\[ rW = (1 - M) [(1 - M)y + M\pi] (u - c) \]

- Note that welfare is increasing in \( \pi \).
Welfare $\pi = 1$

- Note:
  \[ rW = (1 - M) \left( \frac{(1 - M)y + M}{2} \right) (u - c) \]

- Maximize $W$ with respect to $M$:
  \[
  M^* = \begin{cases} 
  \frac{1 - 2y}{2 - 2y} & \text{if } y < \frac{1}{2} \\
  0 & \text{if } y \geq \frac{1}{2} 
  \end{cases}
  \]

- Intuition: facilitate trade versus crowding out barter.
Welfare \( \pi = 0 \)

• Note:

\[ rW = (1 - M) [(1 - M) y] (u - c) \]

• Monotonically decreasing in \( M \) \( \Rightarrow M^* = 0 \).

• Result is a little bit silly: it depends on the absence of free disposal of money. Otherwise, welfare is independent of \( M \).
Welfare $\pi$

Define $M$ such that $\pi = 1$,

- Note:

$$rW = (1 - M) [(1 - M)y + M\pi] (u - c)$$

- Monotonically increasing in $M$ in the $[0, M]$ interval.
Welfare as a Function of M

The diagram illustrates the relationship between welfare (rW) and the level of M. Various curves, such as rW_1^S, rW_1^K, rW_π^K, and rW_π^S, are plotted on a graph with M on the x-axis and rW on the y-axis. The points M_K, M_S, and M_bar mark specific levels of M where the curves intersect or change behavior.
Comparison with Alternative Arrangements

• Imagine that we have the credit arrangement: “produce for anyone you meet that wants your good.”

• Value function

\[ rV_c = u - c \]

• Clearly

\[ rV_c > rW \]

• However, this arrangement is not self-enforceable: histories are not observed.
Second Generation: Endogenous Prices

- We make the very strong assumption that we exchanged one good for one unit of money.


- We set $y = 0$ and we let goods be divisible.

- When agents meet, they bargain about how much $q$ will be exchanged, or equivalently, about price $1/q$. 
Utility and Cost Functions

• Utility is \( u(q) \) and cost of production is \( c(q) \).

• Assumptions:

\[
\begin{align*}
    u(0) &= c(0) = 0 \\
    u'(0) &> c'(0) \\
    u'(0) > 0, u''(0) &\leq 0 \\
    c'(0) > 0, c''(0) &\geq 0
\end{align*}
\]

• Also, \( \hat{q} \) and \( q^* \) are such that

\[
\begin{align*}
    u(\hat{q}) &= c(\hat{q}) \\
    u'(\hat{q}) &= c'(\hat{q}) \\
    u'(q^*) &= c'(q^*)
\end{align*}
\]
Value Functions and Bargaining

• Take \( q = Q \) as given. Then:

\[
\begin{align*}
rV_1 &= (1 - M) [u(Q) + V_0 - V_1] \\
rV_0 &= M [V_1 - V_0 - c(Q)]
\end{align*}
\]

• Bargaining is the generalized Nash bargaining solution:

\[
q = \text{argmax} \left[ u(q) + V_0(Q) - T_1]^{\theta} \times [V_1(Q) - c(q) - T_0]^{\theta}
\right]
\]

\[
\begin{align*}
u(q) + V_0 &\geq V_1 \\
V_1 - c(q) &\geq V_0
\end{align*}
\]

where \( T_j \) is the threat point of the agent with \( j \) units of money.

• We will set \( T_j = 0 \) and \( \theta = 1/2 \).
Equilibria

• Necessary condition taking $V_0(Q)$ and $V_1(Q)$ as given:

$$[V_1(Q) - c(q)]u'(q) = [u(q) + V_0(Q)]c'(q)$$

• The bargaining solution defines a function $q = e(Q)$ and we look at its fixed points.

• Two fixed points:

1. $q = 0$: nonmonetary equilibrium.

2. $q = q^e > 0$: monetary equilibrium.
Monetary Equilibrium in the Divisible-Goods Model

\[ q \]

\[ Q \]

\[ f(Q) \]

\[ e(Q) \]

\[ g(Q) \]

\[ q^e \]

\[ q^* \]

\[ q_1 \]

\[ q \]
Efficiency

• Note that the efficient outcome is $q^*$, i.e. $u'(q^*) = c'(q^*)$.

• In the monetary equilibrium:

$$u'(q^e) = \frac{u(q^e) + V_0(q^e)}{V_1(q^e) - c(q^e)}c'(q^e) > u'(q^*)$$

since $u(q^e) + V_0(q^e) > V_1(q^e) - c(q^e)$.

• Hence $q^* > q^e$, or equivalently, the price is too high.
Third Generation: Endogenous Prices and Goods

- Relax the assumption that agents hold 0 or 1 units of money.

- Problem: endogenous distribution of money that we (and the agents!) need to keep track of.


- Theoretical: