Lecture 5: Lucas Model and Asset Pricing

Economics 714, Spring 2016

1 GE: Arrow-Debreu Complete Markets Model

Each household solves single optimization problem. Representative first order condition:

\[(\beta_i)^t u_i(c_i(s^t)) P(s^t|s_0) = \mu^i q_0^t(s^t)\]

Cross sectionally:

\[(\beta_i)^t u_i(c_i(s^t)) = \mu^i \mu^j\]

2 Asset Pricing

2.1 Lucas (1978) Asset Pricing Model

Large number of identical agents, single nonstorable consumption good (fruit), given off by productive units (trees) with net supply of 1.

Preferences satisfy usual conditions

Owner of tree receives stochastic dividend \(s_t\) with transition function \(Q(s, ds')\).

Representative agent problem:

\[\max_{\{c_t, a_{t+1}\}} E_0 \sum_{t=0}^{\infty} \beta^t u(c_t)\]

subject to:

\[c_t + p_t a_{t+1} = (p_t + s_t) a_t\]
Conjecture pricing function $p_t = p(s_t)$. Then can write Bellman equation:

$$v(a,s) = \max_{(c,a')} \left\{ u(c) + \beta \int v(a',s')Q(s,ds') \right\}$$

subject to:

$$c + p(s)a' \leq (p(s) + s)a, \quad c \geq 0, \quad 0 \leq a' \leq 1$$

## 2.2 Equilibrium

**Definition** A recursive competitive equilibrium is a continuous function $p(s)$ and a continuous, bounded function $v(a,s)$ such that:

1. $v(a,s)$ solves the Bellman equation
2. $\forall s, \ v(1, s)$ is attained by $c = s, \ a' = 1$.

To characterize, note that wealth on hand is what really matters $(p(s) + s)a$. Re-write Bellman:

$$v((p(s) + s)a) = \max_{(a')} \left\{ u((p(s) + s)a - p(s)a') + \beta \int v((p(s') + s')a')Q(s,ds') \right\}$$

First order condition:

$$-u'(c(s))p(s) + \beta \int v'((p(s') + s')a')[p(s') + s']Q(s,ds') = 0$$

Envelope condition:

$$v'((p(s) + s)a) = u'(c(s))$$

Combine to get Euler equation:

$$u'(c(s)) = \beta \int u'(c(s')) \frac{p(s') + s'}{p(s)}Q(s,ds')$$
Or, if $p_t = p(s_t)$, $R_{t+1} = \frac{p_{t+1} + s_{t+1}}{p_t}$:

$$u'(c_t) = \beta E_t[u'(c_{t+1})R_{t+1}]$$

But now this equation determines equilibrium pricing function. In equilibrium $a = a' = 1$, $c(s) = s$:

$$p(s) = \beta \int \frac{u'(s')(p(s') + s')}{u'(s)} Q(s, ds')$$

### 2.3 Pricing General Claims

Say transition function has density $f(s, s')$:

$$Q(s, s') = \int_{-\infty}^{s'} f(s, u) du$$

Pricing kernel $q(s', s)$: equilibrium price $P^A$ of unit of period $t + 1$ when $s_{t+1} \in A$ conditional on $s_t = s$ is

$$P^A(s) = \int_{s' \in A} q(s, s') ds'$$

Here we have:

$$q(s, s') = \beta \frac{u'(s')}{u'(s)} f(s, s')$$

Price of any contingent claim $g(s')$ one-period ahead:

$$p^g(s) = \int q(s, s') g(s') ds'$$

$$= \int \beta \frac{u'(s')}{u'(s)} g(s') f(s, s') ds'$$

$$= \beta \frac{u'(s')}{u'(s)} g(s')$$

$$= \mathbb{E} \left[ \beta \frac{u'(s')}{u'(s)} g(s') \mid s \right]$$

Component $m = \beta \frac{u'(s')}{u'(s)}$ is called stochastic discount factor
For multi-period claims, can chain together one-step claims:

\[ q^j(s, s^j) = \int q(s, s')q^{j-1}(s', s^j)ds' \]

Ownership of tree is claim to entire \( \{s_t\} \) so can define price as:

\[
p(s) = \beta \int \frac{u'(s')}{u'(s)} f(s, s')ds' + \beta^2 \int \frac{u''(s'')}{u'(s)} s'' f^2(s, s'')ds'' + \ldots
\]

Or in sequence notation:

\[
p_t = E_t \left[ \sum_{j=1}^{\infty} \beta^j \frac{u'(s_{t+j})}{u'(s_t)} s_{t+j} \right]
\]

Infinite horizon equilibrium rules out bubbles.