Lecture 4: Consumption-Savings Problem

Under Uncertainty

Economics 714, Spring 2016

1 Consumption-Savings Problem under Uncertainty

1.1 Basic Problem

$$\max_{\{c_t, a_{t+1}\}} E_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$

s.t. $c_t + a_{t+1} = Ra_t + y_t$

$a_0, y_0$ given.

Constraints: $c_t \geq 0, \ a_t \geq a$. Debt limit.

Income $y$ stochastic: $y \in Y \subseteq \mathbb{R}_+$ compact, Borel $\sigma$-algebra $\mathcal{Y}$.

$y$ follows Markov process w/transition function $Q$ on $(Y, \mathcal{Y})$.

Assume $Q$ has **Feller property**: for $f : Y \to \mathbb{R}$ bounded, continuous then:

$$E[f(y')|y] = \int f(y') Q(y, dy')$$

is bounded and continuous

State space: assets $a \in A = [a, \infty)$, Borel $\sigma$-algebra $\mathcal{A}$.

Joint state space $(a, y) \in X = A \times Y$ with Borel $\sigma$-algebra $\mathcal{X}$.

Feasible correspondence:

$$\Gamma(x) = \{(c, a') : c + a' \leq Ra + y, \ c \geq 0, \ a' \geq a\}$$
1.2 Sequence Problem

At each date \( c_t : Y^t \rightarrow \mathbb{R}_+ \), measurable (w.r.t \( Y^t \)).

\( a_{t+1} : Y^t \rightarrow \mathbb{R}_+ \), measurable (w.r.t \( Y^t \))

\( v^*(a_0, y_0) = \sup_{\{c_t(Y^t), a_{t+1}(Y^t)\}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t) \]
\( = \sup_{\{c_t(Y^t), a_{t+1}(Y^t)\}} \int_{\sum_{t=0}^{\infty} \beta^t u(c_t(Y^t)) Q(y_t, dy_{t+1}) Q(y_{t-1}, dy_t) \cdots Q(y_0, dy_1)} \)

1.3 Bellman Equation

\( v(a, y) = \max_{(c,a') \in \Gamma(a,y)} \left\{ u(c) + \beta \int v(a', y') Q(y, dy') \right\} \)

Extensions of the previous results apply, principle of optimality is direct.

Define Bellman operator as before:

\( T f(a, y) = \max_{(c,a') \in \Gamma(a,y)} \left\{ u(c) + \beta \int f(a', y') Q(y, dy') \right\} \)

**Theorem:** Under the assumptions here \( T : C(X) \rightarrow C(X) \) is a contraction, and hence has a unique fixed point \( v \in C(X) \) and for all \( v_0 \in C(X) \):

\( \| T^n v_0 - v \| \leq \beta^n \| v_0 - v \| \)

Moreover, the optimal policy correspondence:

\( G(x) = \{(c, a') \in \Gamma(a, y) : v(a, y) = u(c) + \beta \int v(a', y') Q(y, dy')\} \)

is compact-valued and uhc.

In addition, under our standing assumptions we have the stronger results.
**Theorem:** (i) \( v(a, y) \) is strictly increasing in \( a \)

(ii) \( v(a, y) \) is strictly concave in \( a \) and the optimal policy functions \( c(a, y) \) and \( a'(a, y) \) are continuous.

(iii) If \( (a_0, y_0) \in \text{int}(X) \) and \( (c(a_0, y_0), a'(a_0, y_0)) \in \text{int}\Gamma(a_0, y_0) \) then \( v \) is continuously differentiable (in \( a \)) at \( (a_0, y_0) \) and:

\[
v_a(a_0, y_0) = Ru'(c(a_0, y_0)) = Ru'(Ra + y - a'(a_0, y_0))
\]

### 1.4 Euler Equations

If the constraints don’t bind:

\[
v(a, y) = \max_{(c,a') \in \Gamma(a,y)} \left\{ u(c) + \beta \int v(a', y') Q(y, dy') \right\}
\]

First order condition:

\[
u'(c) = \beta \int v_a(a', y') Q(y, dy')
\]

Envelope condition:

\[
v_a(a, y) = Ru'(c)
\]

So we have the (stochastic) Euler equation:

\[
u'(c(a, y)) = \beta R \int u'(c'(a'(a, y), y')) Q(y, dy')
\]

Or:

\[
u'(c_t) = \beta RE_t u'(c_{t+1})
\]
2 GE: Arrow-Debreu Complete Markets Model

Markov state $s_t$, assume finite with transition function $P(s'|s)$.

Implies probabilities of any sequence $s^t = \{s_0, \ldots, s^t\}$:

$$P(s^t|s_0) = P(s_t|s_{t-1})P(s_{t-1}|s_{t-2})\ldots P(s_1|s_0)$$

Agents: $i = 1, \ldots I$. Endowments: $y_i^t = y^t(s_t)$.

Preferences:

$$U^i(c^t) = \sum_{t=0}^{\infty} \sum_{s^t} (\beta_i)^t u^t(c^t_i(s^t))P(s^t|s^0)$$

Arrow-Debreu complete markets: trade dated, state-contingent consumption claims at date 0, price $q_0^0(s^t))$

Budget constraint:

$$\sum_{s^t} q_0^0(s^t)c^t_i(s^t) \leq \sum_{t=0}^{\infty} \sum_{s^t} q_0^0(s^t)y^t(s_t)$$

Feasible allocation:

$$\sum_{i=1}^{I} c^t_i(s^t) \leq \sum_{i=1}^{I} y^t(s_t), \quad \forall s^t.$$  

A competitive equilibrium is a price system $\{q_0^0(s^t)\}$ and an allocation $\{c^t_i(s^t)\}$ s.t.:

(i) Households optimize, and (ii) markets clear.

Each household solves single optimization problem. Representative first order condition:

$$(\beta_i)^t u^t_i(c^t_i(s^t))P(s^t|s_0) = \mu^i q_0^0(s^t)$$

Cross sectionally:

$$\frac{(\beta_i)^t u^t_i(c^t_i(s^t))}{(\beta_j)^t u^t_j(c^t_j(s^t))} = \frac{\mu^i}{\mu^j}$$