Problem Set 6: Solutions
ECON 301: Intermediate Microeconomics
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Problem 1 (Standard Edgeworth Box)

(a) The total resources in this economy are

MP3s: \( \omega_1 = \omega^E_1 + \omega^M_1 = 10 + 90 = 100 \)

DVDs: \( \omega_2 = \omega^E_2 + \omega^M_2 = 10 + 0 = 10 \)

(b) Allocation \( \omega \) is shown in the Edgeworth box below:

To determine whether the initial allocation is efficient, we need to check whether \( MRS^E(\omega^E) = MRS^M(\omega^M) \). Here \( MRS^i(x^i_1, x^i_2) = -\frac{x^i_2}{5x^i_1} \), so

\[
MRS^E(\omega^E) = -\frac{\omega^E_2}{5\omega^E_1} = -\frac{10}{5 \cdot 10} = -\frac{1}{5}
\]

and

\[
MRS^M(\omega^M) = -\frac{\omega^M_2}{5\omega^M_1} = -\frac{0}{5 \cdot 90} = 0.
\]

Since \( MRS^E(\omega^E) \neq MRS^M(\omega^M) \), the endowment is not Pareto efficient (notice how their
indifference curves intersect). As we’ll see next, it is off the contract curve.

(c) The contract curve is characterized by all the Pareto efficient points, i.e. points for which

$$MRS^M(x_1^M, x_2^M) = MRS^E(x_1^E, x_2^E)$$

and

$$x_1^M + x_1^E = \omega_1 \text{ and } x_2^M + x_2^E = \omega_2$$

(the allocation is feasible and all resources are used—another way of saying that we’re at some point in the Edgeworth box).

From equation (3),

$$-\frac{x_2^M}{5x_1^M} = -\frac{x_2^E}{5x_1^E}.$$  

Since $x_1^E = 100 - x_1^M$ and $x_2^E = 10 - x_2^M$ (from (4)), we can rewrite (5) as follows and solve for $x_2^M$:

$$-\frac{x_2^M}{5x_1^M} = -\frac{10 - x_2^M}{5(100 - x_1^M)} \implies x_2^M = \frac{1}{10} x_1^M$$

after some algebra. So the contract curve is characterized by $x_2^M = \frac{1}{10} x_1^M$, which represents all the Pareto efficient points in the Edgeworth box. Along this line, the indifference curves of Elvis and Miriam are tangent to each other.

(Note: You also could have solved for $x_2^E$ instead and get an equation representing the same line in the Edgeworth box.)

(d) When we find the equilibrium consumption with Elvis and Miriam trading, graphically we’re determining the point in the Edgeworth box for which $MRS^E = MRS^M$ (indifference
curves are tangent to each other, so we’ll be on the contract curve) as well as relative prices $p_1$ and $p_2$ such that the budget line passes through the endowment point and is tangent to $MRS^i$ at the equilibrium point.

The equilibrium in this market is the allocation $(x_1^E, x_2^E)$ and $(x_1^M, x_2^M)$ with prices $(p_1, p_2)$ that satisfy

- **Condition 1**: For both consumers $i = E, M$, $(x_1^i, x_2^i)$ is optimal given prices $(p_1, p_2)$
- **Condition 2**: At prices $(p_1, p_2)$ market clear. This means

$$x_1^E + x_1^M = \omega_1 \quad \text{and} \quad x_2^E + x_2^M = \omega_2.$$ 

We’ll take the following steps to find the equilibrium allocation:

**Step 1: Normalize $p_2$.** Equilibrium determines relative prices, so we can always normalize at least one of the prices to be some constant. The easiest is to just let $p_2 = 1$. (Now we already have one of the six components we need for the equilibrium!) Thinking graphically, the slope of the budget line is $-\frac{p_1}{p_2}$, so any multiple $a$ of $p_1$ and $p_2$ work since $-\frac{p_1}{p_2} = -\frac{ap_1}{a \cdot p_2}$.

**Step 2: Find incomes $m^i$.** In the next step, well find Elvis’s and Miriam’s demand functions, but first we need to know what their incomes are given their endowments (in terms of $p_1$ since we haven’t found that yet.) We have that (recalling that $p_2 = 1$):

$$m^E = p_1 \omega_1^E + p_2 \omega_2^E = 10p_1 + 10$$
$$m^M = p_1 \omega_1^M + p_2 \omega_2^M = 90p_1.$$ 

**Step 3: Find demand functions $x_1^i$.** For this we use the “magic formula” for demand for Cobb-Douglas type utility, $x_1^i = \frac{a \cdot m^i}{a + b \cdot p_1}$. (Of course, if we were dealing with any other type of utility we could derive demand just as we did earlier in the course.) This gives us:

$$x_1^E = \frac{1}{6} \left( \frac{10p_1 + 10}{p_1} \right)$$
$$x_1^M = \frac{1}{6} \left( \frac{90p_1}{p_1} \right)$$

We only really need to get the demand functions for good 1 since we can figure out what $x_2^E$ and $x_2^M$ are from the market clearing conditions, which we’ll use next.

**Step 4: Solve for $p_1$ using demand functions and market clearing conditions.** Plugging the demand functions we just found for good 1 into the market clearing condition for good 1 we
get:
\[ x^E_1 + x^M_1 = \omega_1 \implies \frac{1}{6} \left( \frac{10p_1 + 10}{p_1} \right) + \frac{1}{6} \left( \frac{90p_1}{p_1} \right) = 100 \]

and solving this for \( p_1 \) we get
\[ p_1 = \frac{1}{50}. \]

**Step 5: Use \( p_1 \) and to get optimal consumption.** Now, we can plug \( p_1 = \frac{1}{50} \) into the demand function we found in Step 3 to get optimal consumption levels
\[ x^E_1 = \frac{1}{6} \left( \frac{10p_1 + 10}{p_1} \right) = 85 \]
\[ x^M_1 = \frac{1}{6} \left( \frac{90p_1}{p_1} \right) = 15 \]

We’re almost there. We just need to plug in \( p_1 = \frac{1}{50} \) and \( p_2 = 1 \) into the demand functions for Good 2 now (again, these demand functions come from the “magic formula” for demand with Cobb-Douglas utility):
\[ x^E_2 = \frac{5}{6} \left( \frac{10p_1 + 10}{p_2} \right) = 8.5 \]
\[ x^M_2 = \frac{5}{6} \left( \frac{90p_1}{p_2} \right) = 1.5 \]

So (finally!) we have found equilibrium allocation \( x^E = (85, 8.5), x^M = (15, 1.5) \) and prices \( (p_1, p_2) = (\frac{1}{50}, 1) \).

(e) Any pair of prices that give us the same ratio \( \frac{p_1}{p_2} = \frac{1/50}{1} = \frac{1}{50} \) would work. So, for
instance, prices \((p_1, p_2) = (1, 50)\) support this equilibrium. Prices \((p_1, p_2) = (2, 100)\), so do \((p_1, p_2) = (\frac{1}{2}, 25)\), etc.

(f) Yes, the market efficiently allocates resources. To see this, observe that the MRS for Elvis and Miriam are

\[
MRS^E = -\frac{x_2^E}{5x_1^E} = -\frac{8.5}{5 \cdot 85} = -\frac{1}{50} \quad \text{and} \\
MRS^M = -\frac{x_2^M}{5x_1^M} = -\frac{1.5}{5 \cdot 15} = -\frac{1}{50}.
\]

Their indifference curves are tangent, the allocation is Pareto optimal and we are on the contact curve. Notice also that, even though the total resources are fixed, through trade both of them are better off than they were before! (Utility is higher, they are both on higher indifference curves.)

(g) In this case, all the allocations in the Edgeworth box are Pareto efficient (moving away from any point will not be a Pareto improvement; one cannot be made better off without the other being worse off). Now, let’s think about what prices would work in this market. In equilibrium, with these goods being perfect substitutes for both Elvis and Miriam, for any relative price

\[
\frac{p_1}{p_2} < \text{MRS}^i = \frac{1}{5}
\]

both Elvis and Miriam would want to spend all of their income on \(x_1\) (and none on \(x_2\)). This would result in excess demand for good 1 and markets don’t clear. For any relative price

\[
\frac{p_1}{p_2} > \text{MRS}^i = \frac{1}{5}
\]

they would want to buy only \(x_2\) (and no \(x_1\)), which would result in excess demand for good 2. Therefore, it must be that

\[
\frac{p_1}{p_2} = \text{MRS}^i = \frac{1}{5}
\]

for markets to clear. Now, all of the allocation along the budget line going through the endowment with prices such that \(\frac{p_1}{p_2} = \frac{1}{5}\) is an equilibrium allocation.
Problem 2 (Uncertainty and Asset Pricing)

(a) The Edgeworth box and the point corresponding to their initial endowments (shares held) is shown below. The initial endowment cannot be Pareto efficient since \( MRS^J(\omega^J) \neq MRS^B(\omega^B) \): From this utility function we have that \( MRS^i(x_1^i, x_2^i) = -\frac{x_2^i}{x_1^i} \) and so at the endowment point \( MRS^J(100, 0) = -\frac{0}{100} = 0 \) and \( MRS^B(0, 100) = -\frac{100}{0} = -\infty \).

The endowment (where they are not trading shares) is in fact risky because both of them have different levels of consumption in different states of the world (rainy or no rain).
(b) To find this equilibrium, we go through the same steps as above (so for a more thorough explanation see Problem 3). We normalize $p_2 = 1$ first. Then we get both John and Benjamin’s demand for $x_1$, shares of Rainalot Inc.:

$$x_J^1 = \frac{a \cdot m_J}{a + b \cdot p_1} = \frac{1 \cdot p_1 \times 100 + p_2 \times 0}{2 \cdot p_1} = 50$$

$$x_B^1 = \frac{a \cdot m_B}{a + b \cdot p_1} = \frac{1 \cdot p_1 \times 0 + p_2 \times 100}{2 \cdot p_1} = \frac{50}{p_1}$$

Next we use the market clearing condition $x_J^1 + x_B^1 = 100$ to find $p_1$:

$$x_J^1 + x_B^1 = 100 \implies 50 + \frac{50}{p_1} = 100 \implies p_1 = 1$$

Now given prices $p_1 = p_2 = 1$, we have demand

$$x_J^1 = 50$$

$$x_B^1 = \frac{50}{p_1} = 50$$

So all we have left to do now is to find $x_J^2$ and $x_B^2$. We can plug $p_1 = p_2 = 1$ into their demand for $x_2$, shares of HateRain Inc.:

$$x_J^2 = \frac{b \cdot m_J}{a + b \cdot p_2} = \frac{1 \cdot p_1 \times 100 + p_2 \times 0}{2 \cdot p_2} = 50$$

$$x_B^2 = \frac{b \cdot m_B}{a + b \cdot p_2} = \frac{1 \cdot p_1 \times 0 + p_2 \times 100}{2 \cdot p_2} = 50$$

So our equilibrium allocation is described by prices $(p_1, p_2) = (1, 1)$, shares of $x_1$ held $(x_J^1, x_B^1) = (50, 50)$, and shares of $x_2$ held $(x_J^2, x_B^2) = (50, 50)$.

(c) The allocation is efficient since $MRS_J(50, 50) = -1 = MRS_B(50, 50)$. Also, it is not risky since each of them consumes that same amount regardless of the state of the world (rainy or no rain).
Problem 3 (Irving Fisher Determination of Interest Rate)

(a) The allocation corresponding to the initial endowment is shown below. It is not Pareto efficient since $MRS^J(\omega^J) \neq MRS^W(\omega^W)$. From this utility function we have that $MRS^i(x^i_1, x^i_2) = -\frac{x^i_2}{\beta x^i_1}$ and so at the endowment point $MRS^J(0, 1,000) = -\frac{1,000}{\beta} \cdot 0 = -\infty$ and $MRS^W(1,000, 0) = -\beta \cdot 1,000 = 0$.

(b) To find the equilibrium interest rate $r$, we first need to find $p_1$ and $p_2$, since we’ll use that $\frac{p_1}{p_2} = 1 + r$. Our first step is to normalize $p_2 = 1$. Again, the steps toward finding the market equilibrium are the same as in Problems 3 and 4 (see Problem 4 for more detail.)
We can use the “magic formulas” for demand for consumption today for Jane and William:

\[
C^J_1 = \frac{a}{a + b} \frac{m^J}{p_1} = \frac{1}{1 + \beta} \cdot \frac{p_1 \times 0 + p_2 \times 1,000}{p_1} = \frac{2}{3} \cdot \frac{1,000}{p_1} = \frac{2}{3} \cdot 1,000
\]

\[
C^W_1 = \frac{a}{a + b} \frac{m^W}{p_1} = \frac{1}{1 + \beta} \cdot \frac{p_1 \times 1,000 + p_2 \times 0}{p_1} = \frac{2}{3} \cdot 1,000
\]

Next we use market clearing condition \(C^J_1 + C^W_1 = 1,000\) to get \(p_1\):

\[
C^J_1 + C^W_1 = 1,000 \implies \frac{2}{3} \cdot \frac{1,000}{p_1} + \frac{2}{3} \cdot 1,000 = 1,000 \implies p_1 = 2
\]

Now plugging this into demand for \(C_1\) and \(C_2\) for both Jane and William, we get:

\[
C^J_1 = \frac{2}{3} \cdot \frac{1,000}{p_1} = 333\frac{1}{3}
\]

\[
C^W_1 = \frac{2}{3} \cdot 1,000 = 666\frac{2}{3}
\]

\[
C^J_2 = \frac{b}{a + b} \frac{m^J}{p_2} = 333\frac{1}{3}
\]

\[
C^W_2 = \frac{b}{a + b} \frac{m^W}{p_2} = 666\frac{2}{3}
\]

The interest rate \(r\) is such that \(\frac{p_2}{p_1} = 1 + r \implies r = 1 = 100\%

(c) Yes, the equilibrium is Pareto efficient since \(MRS^J(333\frac{1}{3}, 333\frac{1}{3}) = -2 = MRS^W(666\frac{12}{3}, 666\frac{2}{3})\).

(d) When \(\beta = 1\), going through the same steps as in part (b) we get that \(p_1 = 1\) and hence \(r + 1\). First, getting demand for consumption in period 1:

\[
C^J_1 = \frac{a}{a + b} \frac{m^J}{p_1} = \frac{1}{1 + \beta} \cdot \frac{p_1 \times 0 + p_2 \times 1,000}{p_1} = \frac{1}{2} \cdot \frac{1,000}{p_1} = \frac{1}{2} \cdot 1,000
\]

since \(p_2 = 1\) and

\[
C^W_1 = \frac{a}{a + b} \frac{m^W}{p_1} = \frac{1}{1 + \beta} \cdot \frac{p_1 \times 1,000 + p_2 \times 0}{p_1} = \frac{1}{2} \cdot 1,000
\]

Next we use market clearing condition \(C^J_1 + C^W_1 = 1,000\) to get \(p_1\):

\[
C^J_1 + C^W_1 = 1,000 \implies \frac{1}{2} \cdot \frac{1,000}{p_1} + \frac{1}{2} \cdot 1,000 = 1,000 \implies p_1 = 1
\]
which gives us \( r = 0 = 0\% \). Remember that \( \beta \), here the discount factor, measures patience or how today’s consumption is weighted relative to tomorrow’s consumption. With \( \beta = 1 \), they are now indifferent between the two and therefore do not find saving as costly. In order to equilibrate the market interest rate must go down. Interest rate \( r \) is a reflection of \( \beta \).

(e) Now with \( \beta = \frac{1}{2} \) again with Jane’s income tomorrow being 2,000 we’ll get

\[
C_J^1 = \frac{a}{a + b} \frac{m^J}{p_1} = \frac{1}{1 + \beta} \cdot \frac{p_1 \times 0 + p_2 \times 2,000}{p_1} = \frac{2}{3} \cdot \frac{2,000 p_2}{p_1} = \frac{2}{3} \cdot \frac{2,000}{p_1}
\]

\[
C_W^1 = \frac{a}{a + b} \frac{m^W}{p_1} = \frac{1}{1 + \beta} \cdot \frac{p_1 \times 1,000 + p_2 \times 0}{p_1} = \frac{2}{3} \cdot \frac{1,000}{p_1}
\]

Using market clearing condition \( C_J^1 + C_W^1 = 1,000 \) to get \( p_1 \):

\[
C_J^1 + C_W^1 = 1,000 \implies \frac{2}{3} \cdot \frac{2,000}{p_1} + \frac{2}{3} \cdot \frac{1,000}{p_1} = 1,000 \implies p_1 = 4
\]

Now since \( p_1 = 4 \) and we normalize \( p_2 = 1 \), we’ll get \( \frac{p_1}{p_2} = 1 + r \implies r = 3 = 300\%. \) The intuition is that Jane has a larger endowment tomorrow and she is now able and wants to borrow more. In order to equilibrate the savings market interest rate must go up. This partially reduces her willingness to borrow (relative to a lower interest rate) and also encourages William to lend more.