Arbitrage, Present Values, and Bond Prices

\[ P_t = \frac{100}{1 + i_t} \]  \hspace{1cm} (15.1)
\[ P_2 = \frac{100}{(1 + i_t)(1 + i_{t+1})} \]  \hspace{1cm} (15.2)

If both offer the same one-year return (by arbitrage), then:

\[ 1 + i_t = \frac{P_{t+1}^e}{P_t} \]  \hspace{1cm} (15.3)

Rearranging:

\[ P_t = \frac{P_{t+1}^e}{1 + i_t} \]  \hspace{1cm} (15.4)

What is the numerator of the right hand side of (15.4)? Iterating (15.1) forward, and taking expectations:

\[ P_{t+1}^e = \frac{100}{1 + i_{t+1}} \]

This can be substituted into (15.4) to obtain:

\[ P_t = \frac{100}{(1 + i_{t+1})(1 + i_t)} \]  \hspace{1cm} (15.5)

Bond Prices and Bond Yields, and the Yield Curve

\[ P_t = \frac{100}{(1 + i_t)^2} \]  \hspace{1cm} (15.6)

What will set (15.5) equal to (15.6)?

\[ \frac{100}{(1 + i_{t+1})(1 + i_t)} = \frac{100}{(1 + i_{t+1})^2} \]

Which implies:

\[ (1 + i_{t+1})^2 = (1 + i_{t+1})(1 + i_t) \]
\[ 1 + 2i_{t+1} + i_t^2 = 1 + i_{t+1}i_t + i_{t+1}^2i_t \]
\[ 2i_{t+1} \approx i_{t+1}^2 + i_t \]
\[ \frac{i_{t+1}}{i_t} \approx \frac{1}{2} \left( i_{t+1} + i_t \right) \]  \hspace{1cm} (15.7)
\[ i_{t+1} = 2i_t - i_t \]  \hspace{1cm} (15.8)
Stock Prices

The price of a stock is the present discounted value of the stream of dividends:

\[ S_Q = \frac{D_{t+1}^e}{1 + i_t} + \frac{D_{t+2}^e}{(1 + i_t)(1 + i_{t+1})} + \ldots \]  

(15.9)

Notice by the arbitrage relationship between stocks and bonds:

<table>
<thead>
<tr>
<th>Year t</th>
<th>Year t + 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-year bonds</td>
<td>$1</td>
</tr>
<tr>
<td><strong>Stocks</strong></td>
<td>$1 ( (1 + i_t) )</td>
</tr>
</tbody>
</table>

\[ 1 + i_t = \frac{(D_{t+1}^e + S_{Q_t}^e)}{S_Q} \]

Rearrange:

\[ S_Q = \frac{D_{t+1}^e}{(1 + i_t)} + \frac{S_{Q_t}^e}{(1 + i_t)} \]

(15.A1)

In other words, the stock price today is a function of the expected price tomorrow. Notice by iteration what the expected stock price is, using (15.A1):

\[ S_{Q_t}^e = \frac{D_{t+2}^e}{(1 + i_{t+1})} + \frac{S_{Q_{t+1}}^e}{(1 + i_{t+1})} \]

Substituting into (15.A1) yields:

\[ S_Q = \frac{D_{t+1}^e}{(1 + i_t)} + \frac{D_{t+2}^e}{(1 + i_t)(1 + i_{t+1})} + \frac{S_{Q_{t+2}}^e}{(1 + i_t)(1 + i_{t+1})} \]

Continuing this process leads to:

\[ S_Q = \frac{D_{t+1}^e}{(1 + i_t)} + \frac{D_{t+2}^e}{(1 + i_t)(1 + i_{t+1})} + \ldots + \frac{D_{t+n}^e}{(1 + i_t)(1 + i_{t+1})\ldots(1 + i_{t+n-1})} + \frac{S_{Q_{t+n}}^e}{(1 + i_t)(1 + i_{t+1})\ldots(1 + i_{t+n-1})} \]

(15.A2)

Does the last term on the right-hand side of (15.A2) disappear as \( n \to \infty \). It should in general. Assume interest rates are constant:

\[ \frac{S_{Q_{t+n}}^e}{(1 + i_t)(1 + i_{t+1})\ldots(1 + i_{t+n-1})} = \frac{S_{Q_t}^e}{(1 + i)^n} = \frac{S_{\bar{Q}}}{(1 + i)^n} \]

The last term on the right hand side goes to zero, if the stock prices converges to a constant. Of course, one can’t rule out the stock price going to infinity at a sufficiently fast pace so that the term in the box goes to zero. If that term does not go to zero, then that means there is a (rational stochastic) bubble in stock prices.

The expression (15.9) can be expressed in real terms as well. Divide by the price level, as in Chapter 14 (or see Appendix to Chapter 15):

\[ Q = \frac{D_{t+1}^e}{1 + r_t} + \frac{D_{t+2}^e}{(1 + r_t)(1 + r_{t+1})} + \ldots \]

(15.10)