Algebra for the Growth Accounting Formula

Suppose that each time period $A$, $N$, $K$ grow by $\Delta A / A, \Delta N / N, \Delta K, K$. Further assume:

$$Y = F(N, K, A) = Af(N, K)$$

Take the total differential with respect to time $t$, set $\Delta t = 1$, and manipulating yields:

$$\frac{\Delta Y}{Y} = \frac{\Delta A}{A} + \frac{\Delta f(N, K)}{f(N, K)} \quad (5.6)$$

The last term in (5.6) is a differential with respect to the two arguments of the function, $N$ and $K$. The differential of $f(.)$ with respect to $N$ is the marginal productivity of labor; the differential of $f(.)$ with respect to $K$ is the marginal productivity of capital. Let these be denoted by $M_N$ and $M_K$ respectively, and substitute these expressions into (5.6):

$$\frac{\Delta f(N, K)}{f(N, K)} = M_N \frac{\Delta N}{Y} + M_K \frac{\Delta K}{Y} \quad (5.7)$$

When there are constant returns to scale due to the assumption of a Cobb-Douglas production function.) Substituting (5.7) into (5.6) yields:

$$\frac{\Delta Y}{Y} = \frac{\Delta A}{A} + M_N \frac{\Delta N}{Y} + M_K \frac{\Delta K}{Y} \quad (5.8)$$

Suppose (as we assumed in Chapter 11 under perfect competition), that firms hire factors of production up to the point where the marginal product of a factor equals the real factor price (e.g., capital is hired up to the point where rental cost of capital equals the marginal product of capital), viz.

$$M_N = \frac{W}{P}, M_K = \frac{R^K}{P} \quad (5.9)$$

Substituting into (5.8) yields:

$$\frac{\Delta Y}{Y} = \frac{\Delta A}{A} + \left( \frac{W}{P} \right) \frac{\Delta N}{Y} + \left( \frac{R^K}{P} \right) \frac{\Delta K}{Y} \quad (5.10)$$

Which can be rewritten as:
\[ \Delta Y / Y = \Delta A / A + \left( \frac{WN}{PY} \right) \frac{\Delta N}{N} + \left( \frac{R^K K}{PY} \right) \frac{\Delta K}{K} \]  \hspace{1cm} (5.11) \]

Note that \((WN/PY)\) is the labor share of income, and \((R^K K/PY)\) is the capital share of output. These are measurable; in the United States, the former is 0.7, and the latter is 0.3, \textit{on average}.

Substituting these values into (5.11) leads to equation (5.1) in the text, (5.12) in the appendix.

\[ \Delta Y / Y = \Delta A / A + 0.7 \left[ \frac{\Delta N}{N} \right] + 0.3 \left[ \frac{\Delta K}{K} \right] \]  \hspace{1cm} (5.12) \]

One can re-express (5.12) in terms of output per unit labor \((Y/N)\) by subtracting the growth rate of labor from both sides:

\[ (\Delta Y / Y) - (\Delta N / N) = \Delta A / A + 0.7 \times \left[ \frac{\Delta N}{N} \right] + 0.3 \times \left[ \frac{\Delta K}{K} \right] - (\Delta N / N) \]

Which leads to:

\[ (\Delta Y / Y) - (\Delta N / N) = \Delta A / A + 0.3 \times \left[ \frac{\Delta K}{K} - \frac{\Delta N}{N} \right] \]

Where the left hand side is the growth rate of labor productivity and the term in the [square brackets] on the right hand side is the growth rate of capital per unit labor.

This expression indicates that output per hour can rise because of improving technology (the A term) or because of capital deepening.

\textbf{Figure 1}

\textbf{Labor productivity (year-over-year growth)}