Last week, we began relaxing the assumptions of the symmetric independent private values model. We examined private-value auctions with risk-averse bidders, and showed that the first-price (sealed-bid) auction led to higher expected revenue than the second-price (or ascending) auction. Today, we return to risk-neutral bidders, but remove the assumption of symmetry – that is, we allow for auctions where the bidders’ values are drawn from different probability distributions. For simplicity, we will focus on the case of two bidders, one “strong” and one “weak,” which we will define.

Most of today’s results come from the Maskin and Riley paper, “Asymmetric Auctions.” However, their paper kind of drowns you in notation. One of the nice things about the Milgrom book is that he simplifies the proofs of some of the results, so we’ll be borrowing some proofs from there.

Today’s main results:

- Strong bidders (bidders whose \( F_i \) favor higher values) prefer second-price (or ascending) auctions
- Weak bidders prefer first-price (sealed-bid) auctions
- (This holds both ex-ante, and once the bidders have learned their types)
- The seller’s preferences can go either way

The general setup: two risk-neutral bidders, one named Strong and one named Weak, with independent private values drawn from \( F_S \) and \( F_W \). We will generally assume that \( F_S \) first-order stochastically dominates \( F_W \), which we showed earlier was equivalent to \( F_S(t) \leq F_W(t) \) for every \( t \). Actually, we’ll assume that this holds strictly at all interior values of \( t \), plus something a bit stronger.

(Like before, what we’re really trying to do is to compare first-price sealed-bid auctions to ascending (English) auctions; but in this setting, that’s the same as comparing them to second-price auctions.)

In a second-price auction with asymmetric bidders, it is still a dominant strategy to bid your value; so the initial focus is on understanding equilibrium bid strategies in the asymmetric first-price auction.

Maskin and Riley cite an earlier paper establishing that in general, an asymmetric two-bidder first-price auction has a unique equilibrium, which is a solution to the differential equations defined by the bidders’ first-order conditions, subject to the appropriate boundary conditions.
conditions. If we let \( b_S \) and \( b_W \) be the equilibrium bid functions, then the strong bidder’s problem is

\[
\max_b F_W(b_W^{-1}(b))(t_S - b)
\]

or

\[
\max_b \log F_W(b_W^{-1}(b)) + \log(t_S - b)
\]

with first-order condition

\[
\frac{f_W(b_W^{-1}(b))}{F_W(b_W^{-1}(b))} (b_W^{-1})'(b) - \frac{1}{t_S - b}
\]

and, setting this equal to 0 at \( b = b_S(t_S) \), or \( t_S = b_S^{-1}(b) \),

\[
\frac{f_W(b_W^{-1}(b))}{F_W(b_W^{-1}(b))} (b_W^{-1})'(b) = \frac{1}{b_W^{-1}(b) - b}
\]

Similarly, for the weak type,

\[
\frac{f_S(b_S^{-1}(b))}{F_S(b_S^{-1}(b))} (b_S^{-1})'(b) = \frac{1}{b_S^{-1}(b) - b}
\]

In a first-price auction, it is never a best-response to bid strictly higher than your opponent’s highest bid, so if we let the supports of the type distributions be \([t_S, \tilde{t}_S] \) and \([t_W, \tilde{t}_W] \), then \( b_W(\tilde{t}_W) = b_S(\tilde{t}_S) \). As for the boundary condition at the bottom, first-order stochastic dominance implies that \( t_W \leq t_S \). When these are equal, the other boundary condition is \( b_W(t_W) = b_S(t_S) = t_W = t_S \). (It’s easy to show that these are necessary conditions for equilibrium. Along with the differential equations, they turn out to be sufficient as well.) When \( t_W < t_S \), the boundary condition gets a little more complicated, as there may be a mass of weak bidders who knowingly submit losing bids. Later on, we’ll also consider auctions with reserve prices, in which case the relevant boundary condition is \( b_W(r) = b_S(r) = r \) if \( r \) is within the support of both bidders’ type distributions.

We can use the ranking lemma from last week to show that the probability distribution of \( b_S(t_S) \) first-order stochastically dominates the distribution of \( b_W(t_W) \). Recall the ranking lemma: if \( g(x^*) \geq h(x^*) \), and for \( x \geq x^* \), \( g(x) = h(x) \to g'(x) > h'(x) \), then \( g > h \) above \( x^* \). In this case, the boundary condition at the top is cleaner, so we’ll rephrase the ranking lemma to start from the top.

**Lemma 1.** Consider two continuous, differentiable functions \( g \) and \( h \). Suppose \( g(x^*) \geq h(x^*) \), and for \( x \leq x^* \), \( g(x) = h(x) \to g'(x) < h'(x) \). Then \( g > h \) below \( x^* \).

(Proof: see \( g \) and \( h \) as functions of \( x^* - x \), not \( x \). Then the signs of the derivatives flip and satisfy the conditions of our original ranking lemma.)

**Lemma 2.** If the strong bidder’s type distribution strictly FOSD the weak bidder’s, then the strong bidder’s bid distribution strictly FOSD the weak bidder’s.
For $i$ either $S$ or $W$, $\Pr(b_i(t_i) < b) = \Pr(t_i < b_i^{-1}(b)) = F_i(b_i^{-1}(b))$, so what we want to show is

$$F_S(b_S^{-1}(b)) < F_W(b_W^{-1}(b))$$

for $b$ in the interior of the range of the bid functions. As we argued, the bid functions must have the same upper bound, which we will call $b^*$; so $F_S(b_S^{-1}(b^*)) = F_W(b_W^{-1}(b^*)) = 1$. For $b < b^*$, suppose $F_S(b_S^{-1}(b)) = F_W(b_W^{-1}(b))$. By assumption, $F_S(t) < F_W(t)$ (the type distribution of the strong bidder first-order stochastically dominates); and $F_S$ and $F_W$ are increasing, so this requires $b_S^{-1}(b) > b_W^{-1}(b)$. Then from the two bidders’ first-order conditions,

$$f_W(b_W^{-1}(b)) (b_W^{-1})'(b) = \frac{F_W(b_W^{-1}(b))}{b_W^{-1}(b) - b} < \frac{F_S(b_S^{-1}(b))}{b_S^{-1}(b) - b} = f_S(b_S^{-1}(b)) (b_S^{-1})'(b)$$

so we just showed that $F_W(b_W^{-1}(b)) = F_S(b_S^{-1}(b))$ implies

$$\frac{d}{db} F_W(b_W^{-1}(b)) < \frac{d}{db} F_S(b_S^{-1}(b))$$

So the conditions for the ranking lemma are met, and $F_W(b_W^{-1}(b)) > F_S(b_S^{-1}(b))$; or the strong bidder’s bid distribution strictly FOSD the weak bidder’s.

Maskin and Riley make a stronger assumption than FOSD about how the distributions of the two bidders’ values differ. Specifically, they require a condition they call Conditional Stochastic Dominance:

**Assumption 1.** For any $t$ and $t'$ with $t < t'$, $\Pr(t^S < t | t^S < t') < \Pr(t^W < t | t^W < t')$.

(Usual FOSD is simply that this holds for $t' = \bar{t}$ only; so CSD implies FOSD, but is stronger.)

This is equivalent to

$$\frac{F_S(t)}{F_S(t')} < \frac{F_W(t)}{F_W(t')}$$

for any $t < t'$, which (rearranging) is equivalent to

$$\frac{F_S(t)}{F_W(t)} < \frac{F_S(t')}{F_W(t')}$$

for any $t < t'$, or $\frac{F_S}{F_W}$ strictly increasing, that is,

$$\frac{d}{dt} \frac{F_S(t)}{F_W(t)} > 0$$

which (taking the derivative) is equivalent to

$$\frac{1}{(F_W(t))^2} \left( F_W(t) F'_S(t) - F_S(t) F'_W(t) \right) > 0$$

or, dropping the denominator and rearranging,

$$\frac{F'_S(t)}{F_S(t)} > \frac{F'_W(t)}{F_W(t)}$$

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(Milgrom recasts the type distributions as $F(t|s)$, where $s$ takes the value of 0 for the weak bidder and 1 for the strong bidder. And since $F'/F$ is the derivative of $\log F$, he recasts the assumption of Conditional Stochastic Dominance as the requirement that $\frac{d}{ds} \log F$ be increasing in $s$, or $\log F$ is supermodular – i.e., has increasing differences – in $t$ and $s$.)

**Lemma 3.** If $F_S$ CSD $F_W$, then $b_S(t) < b_W(t)$ for interior $t$.

That is, at the same type, a strong bidder shades his bid further below his actual value. (Note that this immediately implies that asymmetric first-price auctions are not efficient – the weak bidder will get the object when his type is slightly below the strong bidder’s.)

First-order stochastic dominance implies $t_W \leq t_S$ (because $F_W(t_S) \geq F_S(t_S) = 1$). Since the bid functions have the same range and $b_S$ is increasing,

$$b_W(t_W) = b_S(t_S) \geq b_S(t_W)$$

Now pick $t < t_W$, and suppose $b_S(t) = b_W(t)$. The two bidders’ first-order conditions are

$$\frac{f_W(t)}{F_W(t)} \frac{1}{b_W'(t)} = \frac{1}{t - b_W(t)} = \frac{1}{t - b_S(t)} = \frac{f_S(t)}{F_S(t)} \frac{1}{b_S'(t)}$$

But Conditional Stochastic Dominance implies $\frac{f_W(t)}{F_W(t)} < \frac{f_S(t)}{F_S(t)}$, so for equality, $b_W'(t) < b_S'(t)$. Which implies (by the ranking lemma) that $b_W(t) > b_S(t)$ for $t$ below $t_W$.

So under the assumption that $F_S$ conditionally first-order stochastically dominates $F_W$, we now know that

- at the same value $t$, the weak bidder bids more aggressively
- but the strong bidder’s bid distribution still first-order stochastically dominates

Now we come to the interesting part:

**Theorem 1.** Suppose $F_S$ conditionally first-order stochastically dominates $F_W$. Then comparing a first-price and second-price auction,

- Every type of strong bidder prefers the second-price auction
- Every type of weak bidder prefers the first-price auction

(For auctions without reserve prices, the preferences are strict for all types above the lowest. For auctions with reserve prices, the preferences are strict for all types above the reserve price. We prove it below for a first- and second-price auctions with the same (binding) reserve price; for auctions without a reserve price, just set $r = 0$.)

**Proof.** First of all, we know that $b_S(t)$ and $b_W(t)$ have the same range, so define $m(t) \equiv b_W^{-1}(b_S(t))$ as the type of weak bidder who bids the same as a strong bidder with type $t$ in the first-price auction. (This function is sometimes referred to as a “matching function” or “tying function”.) Since $b_S(t) < b_W(t)$ for interior $t$, we know that $m(t) < t$ for interior $t$. 


We will calculate the strong bidder’s expected payoff at a type \( t \) using the envelope theorem. By the usual argument,

\[
f^S(x,t) = \Pr(b_W(t_W) < x)(t - x)
\]

Taking the derivative with respect to \( t \) gives

\[
f_t^S(x,t) = \Pr(b_W(t_W) < x)
\]

and evaluating this at \( x = b_S(t) \) gives

\[
f_t^S(b_S(t), t) = \Pr(b_W(t_W) < b_S(t)) = \Pr(t_W < m(t)) = F_W(m(t))
\]

Since given reserve price \( r \), \( V_S(r) = 0 \), by the envelope theorem,

\[
V_{SFP}(t) = \int_r^t F_W(m(s))ds
\]

in a first-price auction with a reserve price \( r \). On the other hand, in a second-price auction, both bidders bid their types, so

\[
f_t^S(b_S(t), t) = \Pr(t_W < t) = F_W(t)
\]

and so

\[
V_{SP}(t) = \int_r^t F_W(s)ds
\]

Since \( m(s) < s \) and \( F_W \) is strictly increasing, the strong bidder strictly prefers the second-price auction for \( t > r \).

By the same exact logic, the weak bidder’s expected payoff is

\[
V_{WFP}(t) = \int_r^t F_S(m^{-1}(s))ds
\]

in the first-price auction, and \( V_{WSP}(t) = \int_r^t F_S(s)ds \) in the second-price auction; since \( m^{-1}(s) > s \), the first is higher.

So we have the nice, fairly general result that strong bidders prefer open auctions and weak bidders prefer sealed-bid auctions.

(My intuition for this is that strong bidders are very likely to value the object more, and therefore very likely to try to outbid whatever the weak bidder does. In an ascending auction, he can see the weak bidder’s bid, so he can just barely outbid him; in a sealed-bid first-price auction, he has to guess, and since he likely values the good so highly, he has to aim near the top of the weak bidder’s bid distribution. Maskin and Riley show the extreme example where \( F_W = U[0, 1] \) and \( F_S = U[2, 3] \). In the second-price auction, the strong bidder always wins and pays \( t_W \), which on average is \( \frac{1}{2} \). In the first-price auction, the equilibrium is for the weak bidder to bid his type and the strong bidder to always bid 1, regardless of type; so the strong bidder still always wins, and pays 1. In this case, the
results above don’t hold – equilibrium isn’t characterized by the first-order conditions – but the intuition goes the same way.)

Maskin and Riley point out that empirically, this seems to have been understood by bidders, at least in some instances. In timber auctions in the Pacific Northwest, local “insiders” have lobbied hard for open auctions. Similarly, buyers who perceived themselves to be “strong” lobbied the FCC to use some form of open auction in the early spectrum auctions. This is also something Klemperer points to as one of the failings of the ill-fated 3G auction in Switzerland, which raised one-thirtieth of the per-capita revenue of the British and German auctions held the same year: Switzerland ran an ascending auction, causing potential entrants who were expected to be “weaker” than the few incumbent firms to give up. (At least one company hired bidding consultants and then chose not to bother bidding.)

**Expected Revenue**

Now we come to the seller’s problem. Consider auctions without reserve prices.

We know from before that the expected revenue of any auction is the expected value of the marginal revenue of the winner. We know that the second-price auction is efficient – gives the object to the bidder with the highest value – which without symmetry need not be the bidder with the highest marginal revenue. We just showed that the asymmetric first-price auction is not efficient – the strong bidder bids lower relative to his true valuation, so when the two bidders’ types are close together, the first-price auction is biased toward giving the object to the weak bidder. Which is better from the seller’s point of view depends on whether the two “distortions” go in the same direction or different directions.

To come up with a general principle... If, conditional on their types being in the same ballpark, the weak bidder is likely to have significantly higher marginal revenue, then first-price auction is probably revenue-superior. If, conditional on their types being in the same ballpark, the weak bidder is likely to have the same or lower marginal revenue, the second-price auction is probably revenue-superior.

Two examples to bear out this intuition. First, suppose the weak bidder’s types are $U[0, 1]$, and the strong bidder’s types are $U[0.5, 1.5]$. Marginal revenue for each is

$$MR_i(t) = t - \frac{1 - F_i(t)}{f_i(t)} = \begin{cases} t - (1 - t) = 2t - 1 & \text{for weak bidder} \\ t - (1.5 - t) = 2t - 1.5 & \text{for strong bidder} \end{cases}$$

So for types that are reasonably close together, the weak bidder has higher marginal revenue than the strong bidder; so the first-price auction, which is skewed towards giving the object to the weak bidder, turns out to be revenue-superior.

Now suppose instead that the strong bidder has types that are $U[0, 1]$, and the weak bidder has types which with probability $\frac{1}{2}$ are $U[0, 1]$, and with probability $\frac{1}{2}$ are 0. The marginal revenue for a strong bidder with type $t$ is $2t - 1$; the marginal revenue for a weak bidder is

$$MR_W(t) = t - \frac{1 - F_i(t)}{f_i(t)} = t - \frac{(1 - t)/2}{1/2} = 2t - 1$$
So in this case, even though bidders are asymmetric, the bidder with the higher type also has the higher marginal revenue. In that case, the second-price auction, which is efficient, is also higher-revenue.

Maskin and Riley give three results on settings where one auction is revenue-superior to another. Unfortunately, in each case, they need to be fairly restrictive about the conditions they impose on the distributions; so interpreting these as robust positive results is a little tricky. On the other hand, it’s easy to interpret them as negative results: that is, that sometimes the first-price auction revenue-dominates, and sometimes the second-price auction revenue-dominates, and that it doesn’t take a particularly pathological example for either one to happen.

**Theorem 2.** Suppose that $F_S$ is just $F_W$ right-shifted by a constant. If $F_W$ is weakly convex and strictly log-concave, and $MR_W(t) \leq 0$ for $t < t_S$, then the sealed-bid auction gives higher expected revenue than the ascending auction.

The requirement that $F_W$ be convex rules out many familiar-looking distributions; but these conditions do allow the uniform distribution, the exponential distribution $F(t) \propto e^{-t}$, and the distribution $F(t) \propto \frac{1}{t}$. The other condition is basically that the right-shift not be too severe. (Our example with $U[0, 1]$ and $U[.5, 1.5]$ fit these requirements. The example with $U[0, 1]$ and $U[2, 3]$ does not, since the equilibrium characterization by differential equations breaks down. In that particular example, the first-price auction still turns out to be revenue-superior, but the same proof doesn’t work and the result may not generalize.)

**Theorem 3.** Suppose that $F_W$ is a truncation of $F_S$, that is, there is some $t^*$ such that $F_W(t) = F_S(t)/F_S(t^*)$. Under some regularity conditions on $F_S$, the sealed-bid auction gives higher expected revenue than the ascending.

**Theorem 4.** Suppose $F_W$ is just $F_S$, with some of the weight taken from the distribution and moved to the lower end. If $F_S$ has an increasing hazard rate, and with an assumption on how the mass is taken, the ascending auction gives higher expected revenue than the sealed-bid auction.