Tuesday, we analyzed first-price auctions in drainage tract models - common-value auctions with one informed bidder and one or more uninformed bidders. Today, we stick with common values.

In the drainage tract model, we found that the first-price auction had a unique equilibrium. (Well, actually unique when $N = 2$, functionally unique when $N \geq 3$.) In the Milgrom Weber paper, when equilibrium strategies were defined for the two types of ascending auctions, however, it was claimed only that these strategies were an equilibrium, not that there were no others. Which brings us to the following problem: with common values, there are often lots of equilibria.

Let's look at a simple example of this: two bidders, with types $t_1$ and $t_2$ which are independently $U[0, 1]$, and

$$v_1(t_1, t_2) = v_2(t_1, t_2) = t_1 + t_2$$

It's not hard to see that each bidder bidding $b_i(t_i) = 2t_i$ is one ex post equilibrium of the second price auction. If my opponent bids $2t_j$, this is the price I have to pay if I want the object, which I value at $t_i + t_j$. So if $t_i + t_j > 2t_j$, or $t_i > t_j$, I want to outbid him, and when $t_i < t_j$, I don’t; bidding $2t_i$ accomplishes this.

In the symmetric equilibrium, the bidder with the higher type wins. (In common-value auctions, there is no question of efficiency, since the bidders value the object the same.) A winning bidder pays on average $E(2t_j|2t_j < 2t_i) = t_i$ for an object worth, on average, $t_i + E(t_j|t_j < t_i) = \frac{3}{2}t_i$. Expected revenue is $E \min\{2t_1, 2t_2\} = \frac{2}{3}$.

Now consider the following alternative strategies:

$$b_1(t_1) = \frac{3}{2}t_1, \quad b_2(t_2) = 3t_2$$

If I’m player 1, I now have to pay $3t_2$ if I win, so I want to win when $t_1 + t_2 > 3t_2$, or $t_1 > 2t_2$, or $\frac{3}{2}t_1 > 3t_2$. So even if I know his bid and type, if player 2 bids $3t_2$, I can’t do better than by bidding $\frac{3}{2}t_1$. Similarly, if 1 bids $\frac{3}{2}t_1$, bidder 2 wants to outbid him whenever $t_1 + t_2 > \frac{3}{2}t_1$, or $t_2 > \frac{1}{2}t_1$, or $3t_2 > \frac{3}{2}t_1$, which he accomplishes by bidding $3t_2$.

So these strategies form another ex post equilibrium. But in this one, bidder 1 wins less often than before, and pays more, on average, when he does win. On the other hand, bidder 2 wins more often, and pays less on average.

One more:

$$b_1(t_1) = \frac{101}{100}t_1, \quad b_2(t_2) = 101t_2$$
The logic is the same; this is indeed an ex post equilibrium. Now bidder 1 wins whenever \( t_1 > 100t_2 \), which is hardly ever, and earns hardly any surplus when he does win. On the other hand, bidder 2 almost always wins, and pays on average just barely more than \( \frac{1}{2} \). And the expected revenue is very close to \( \frac{1}{2} \), lower than before.

These are all examples of a more general result:

**Theorem 1.** Suppose \( N = 2 \) and \( v_1 = v_2 = v(t_1, t_2) \). Then for any continuous and strictly increasing function \( f : \mathbb{R} \to \mathbb{R} \),

\[
  b_1(t_1) = v(t_1, f^{-1}(t_1)), \quad b_2(t_2) = v(f(t_2), t_2)
\]

is an ex post equilibrium.

The proof looks just like the logic we’ve already done. If bidder 2 bids \( v(f(t_2), t_2) \), 1 wants to win whenever

\[
  v(t_1, t_2) > v(f(t_2), t_2)
\]

which is whenever \( t_1 > f(t_2) \), which is whenever \( f^{-1}(t_1) > t_2 \), which is exactly the times when

\[
  v(t_1, f^{-1}(t_1)) > v(f(t_2), t_2)
\]

so there you go.

So what does this mean? That there’s a massive number of equilibria in ascending (or second-price) common-value auctions.

In the example we did before, \( v = t_1 + t_2 \), and bidders’ bids are bounded below by \( t_1 \), so while revenue is lower than in the symmetric auction, it isn’t crazy low. This isn’t always the case, though. Suppose types are within \([0, 1]\), and consider the equilibrium where \( f(s) = s^M \), where \( M \) is large. Suppose instead of being additive, though, types are multiplicative, that is, \( v(t_1, t_2) = t_1 t_2 \). Then

\[
  b_2(t_2) = v(f(t_2), t_2) = t_2^\alpha t_2 = t_2^{1 + \alpha}
\]

As \( \alpha \) gets large, this goes to 0 for all \( t_2 < 1 \). So for \( \alpha \) large, this equilibrium says that bidder 1 always wins, and always pays 0.

Equilibria like this are sometimes referred to as “collusive” equilibria – bidder 2 appears to be letting bidder 1 win. But the strategies still form an equilibrium (and, in fact, and ex post equilibrium).
Almost Common Values

Klemperer’s paper on almost common values makes the case that when one player has even a slight private-value advantage over the other in an otherwise common-values setting, this type of outcome is the only one we should expect in an ascending auction.

He models the wallet game – an independent additive common-values ascending auction like we started today with. Two people look in their wallets, count the money there, and then hold an ascending bid for a prize equal to the combined money in both their wallets. But player 1, if he wins, also gets an extra dollar.

Klemperer’s point is that this small advantage has an explosive effect on the outcome. “Well-behaved” equilibrium play in an ascending auction reveals the loser’s type. Continuing to bid in an ascending, common value auction is predicated on the following logic: “given my own information, and given your equilibrium bid strategy, I’m comfortable paying this price for the object, even if you drop out right now.” Suppose I’m the advantaged player, and I drop out at some point during the auction; equilibrium play implies that knowing that, my opponent doesn’t mind paying the current price for the object. But if that’s true, then since I the object is worth even more to me, I shouldn’t have dropped out.

This means that, if I have even a very small private-value advantage in a common-value ascending auction, I should never lose! Here’s one way to put this formally:

**Theorem 2.** Suppose that $v_1 = v(t_1, t_2) + \epsilon$ and $v_2 = v(t_1, t_2)$. In any equilibrium of an ascending auction with continuous, undominated strategies, bidder 1 wins with probability 1.

Let types be between 0 and 1. Using undominated strategies, bidder 1 given type 1 must bid at least $v(1, 0) + \epsilon$, while bidder 2 given type 0 must bid at most $v(1, 0)$, so there is no such equilibrium where 2 always wins.

Since by assumption bid functions are continuous, if both bidders win with positive probability, then there must be some interval of bids within the range of both bidders’ bid functions. Pick $b$ some bid in the interior of this region, and define $t_1$ and $t_2$ such that

$$b = \beta_1(t_1) = \beta_2(t_2)$$

If $v(t_1, t_2) > b$, then bidder 2 can gain by increasing his bid slightly, since the additional times he wins are profitable. Similarly, if $v(t_1, t_2) < b$, then bidder 2 can gain by decreasing his bid slightly, since he is now winning unprofitably. So $b = v(t_1, t_2)$. But by the same logic, $b = v(t_1, t_2) + \epsilon$. Since this is impossible, there can’t be any range where bid distributions overlap, so bidder 1 always wins.

(The caveat of undominated strategies is important, though. Suppose $v = t_1 + t_2$, and $t_1$ and $t_2$ are between 0 and 1. Bidder 1 always bidding 0 and bidder 2 always bidding 50 is an equilibrium as well; but both bidders are playing weakly-dominated strategies at every stage.)
Klemperer gives several examples of auctions where common values are a reasonable approximation, but one participant is expected to have a slight edge over the others in terms not of information but of the ex-post value of the object:

- The Los Angeles PCS license in one of the spectrum auctions, where one bidder, Pacific Telephone, was already established as a local brand and had a database of potential local customers
- Leveraged Buyout firms competing to acquire a company, but when one of them already has a small ownership stake, or “toehold”, making the company slightly cheaper for it to acquire
- Glaxo bidding against other pharmaceutical companies to acquire Wellcome, where most people felt Glaxo had slightly greater synergies with Wellcome than its competitors

In fact, Pacific Telephone did win the LA PCS license, and at a price many felt was very low. In the Glaxo-Wellcome case, bidders faced bidding costs which were substantial (but dwarfed by the potential gains); Glaxo submitted the first bid, and although two other companies gave serious consideration to bidding higher, neither did, and Glaxo acquired Wellcome at their initial price.

The Milgrom book gives our theorem in a slightly more general version – he requires only that $v_1(t_1, t_2) > v_2(t_1, t_2)$ everywhere, not that they differ by a constant amount $\epsilon$ – and he offers the following as a representative ex post equilibrium: $b_1(t_1) = v_1(t_1, 1)$, and $b_2(t_2) = v_2(0, t_2)$. That is, bidder 1 bids his value if bidder 2 is his best possible type, and bidder 2 bids his value if bidder 1 is his worst possible type. Bidder 1 always wins; and the average revenue is $E_{t_2}v_2(0, t_2)$, which could be pretty low.

Klemperer’s point, then, is that sealed-bid (first-price) auctions are preferable when one bidder has a private-value advantage over another. The advantage does not get magnified in the same way; and so even the disadvantaged bidders earn substantial profit, and are therefore undeterred by small bidding or entry costs.

We showed earlier that in a private-values setting, even when value distributions overlap, “strong” bidders prefer ascending auctions, while “weak” bidders prefer first-price auctions. This is sort of a similar result. This time, values are mostly common, but with a small private component; this time the “strong” bidder always wins, often at a low price, and the “weak” bidder gets 0 expected profits. This is another way to interpret the failed Swiss 3G spectrum auction: once an ascending format was announced, the marginal competitors chose not to participate, leading to an insufficient number of bidders to generate true competition.

Klemperer also points out that in spectrum auctions, there is a gain to holding ascending auctions, since with multiple (possibly complementary) prizes up for auction simultaneously, firms can figure out which ones they may be be able to win before committing too much money to buying complementary ones; he proposes the Anglo-Dutch auction we discussed.
Tuesday as a way to capture much of this benefit without falling prey to low revenues due to private-value differences among firms. He also suggests a solution when selling a company.

**Symmetry**

The drainage tract auction gave us an interesting result: it was a very asymmetric setting, that led to an equilibrium with a strong type of symmetry. (The probability distributions of the informed and uninformed bidders’ equilibrium bids were the same.)

This type of symmetry – Milgrom refers to it as “distribution matching” – occurs in a couple other settings as well.

Consider the following two-bidder common value setting: bidder signals are independently $U[0, 1]$, and the common value $v(t_1, t_2)$ is increasing in both signals. The surprising result: even if $v$ is not symmetric, the first-price auction has a unique equilibrium, in which the two bidders’ bid distributions are the same.

(For example, suppose $v = 100t_1 + t_2$, so bidder 1’s information is vastly more relevant to the value of the object; bidder 1 nonetheless wins exactly half the time in equilibrium.)

Note that we do need the assumption that the two bidders’ signals are independent. However, it is not the case that “all hell breaks loose” if there is a small amount of correlation; Parreiras characterizes equilibrium of two-bidder common-value auctions with affiliated signals, and things are continuous. So with a little bit of correlation, one bidder may win a little bit more often, but there is not some huge discontinuous leap.

Note also that the drainage tract model is simply the limiting case where $v$ does not depend on $t_2$.

To see why this symmetry occurs, suppose that bidder $j$ plays a strictly increasing bid strategy $b_j$. Then given a true type $t_i$, bidder $i$ faces the maximization problem

$$\max_b \Pr(t_j < b_j^{-1}(b)) \left( E(v(t_i, t_j)|t_j < b_j^{-1}(b)) - b \right)$$

If we let $G_j$ be the cumulative distribution of $b_j(t_j)$, we can rewrite this as

$$\int_0^b (v(t_i, b_j^{-1}(s)) - b)dG_j(s)$$

Differentiating with respect to $b$ gives

$$v(t_i, b_j^{-1}(b))g_j(b) - \int_0^b dG_j(s) = v(t_i, b_j^{-1}(b))g_j(b) - G_j(b)$$

In equilibrium, this is equal to 0 at $b = b_i(t_i)$, or $t_i = b_i^{-1}(b)$, so

$$v(b_i^{-1}(b), b_j^{-1}(b))g_j(b) = G_j(b)$$

or

$$\frac{1}{v(b_i^{-1}(b), b_j^{-1}(b))} = \frac{g_j(b)}{G_j(b)}$$
But note that the left-hand side is the same for both bidders; that is, if we re-did the same analysis using $G_i$ the distribution of $b_i(t_i)$ and bidder $j$’s first-order condition, we’d find

$$\frac{1}{v(b_i^{-1}(b), b_j^{-1}(b))} = \frac{g_i(b)}{G_i(b)}$$

and therefore

$$\frac{g_i(b)}{G_i(b)} = \frac{g_j(b)}{G_j(b)}$$

for any $b$ in the range of both bid functions; or

$$\frac{d}{db} \log G_i(b) = \frac{d}{db} \log G_j(b)$$

But by the usual logic, in a first-price auction, equilibrium bid functions must have identical ranges; so this holds at all equilibrium bids of both bidders. And there is some maximum bid $b^*$ that’s the maximum bid for both bidders, so

$$\log G_i(b^*) = \log G_j(b^*) = \log 1 = 0$$

and so, writing $\log G_i$ as the integral of its derivative, $G_i = G_j$ everywhere, that is, the two bidders have the same equilibrium bid distributions. (Since we normalized the types to be drawn from the same distribution, this means that $b_1(s) = b_2(s)$.)

So far, we’ve just shown that this happens in any equilibrium, not that equilibrium is unique. But now we can make some headway with the envelope theorem. Since any equilibrium is symmetric, we can write bidder $i$’s problem as

$$V_i(t_i) = \max_x \int_0^x (v(t_i, s) - b(x)) ds$$

since by bidding $b(x)$, you win whenever $t_j < x$. By the envelope theorem, then,

$$V'_i(t_i) = \int_0^{t_i} v^1(t_i, s) ds$$

and so

$$V(t_i) = \int_0^{t_i} \int_0^t v^1(t, s) dsdt$$

Changing the order of integration gives

$$V(t_i) = \int_0^{t_i} \int_s^{t_i} v^1(t, s) dt ds = \int_0^{t_i} (v(t_i, s) - v(s, s)) ds$$

Of course, we can use our usual $Pr \times (v - b)$ notation to rewrite $V(t_i)$ as

$$V(t_i) = t_i \left( \frac{1}{t_i} \int_0^{t_i} v(t_i, s) ds - b(t_i) \right)$$

and, equating them, find that

$$b(t_i) = \frac{1}{t_i} \int_0^{t_i} v(s, s) ds = E(v(s, s) | s < t_i)$$

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so equilibrium strategies are uniquely pinned down.

In one of my papers, I assume that $v = f_1(t_1) + f_2(t_2)$. In that case, bidder $i$’s value function does not depend on $f_2$, and expected payoffs are exactly like those of the informed bidder in the drainage tract model, as a function of only his own information. This allows us to extend the information results from the drainage tract model to general two-bidder common-value auctions with independent signals.

**Asymmetric Private-Value Auctions with Resale**

Independent signals and common values, however, may be a little bit hard to justify. However, Hafalir and Krishna generate exactly the same symmetry result in perhaps a more natural setting. They point out (as we’ve seen) that first-price private-value auctions with asymmetric bidders are not necessarily efficient, that is, the loser sometimes values the object more than the winner. This allows the possibility of profitable resale (by the winner to the loser) after the auction ends.

They assume two asymmetric bidders; a first-price auction is held, the loser’s bid is not revealed, and the winner can then make a take-it-or-leave-it offer to sell the good to the loser at some price. The loser accepts or rejects the offer, and the game ends.

Note that while the winning bid may or may not be revealed prior to the resale stage, it is important that the losing bid is not. This is because the loser’s bid generally reveals his type, allowing the winner to charge him his full value at the resale stage, which would change the game.

Their main result is that when resale is a possibility, the seller prefers a first-price auction to a second-price auction. However, one of the interesting results along the way is exactly what we just saw – that the two bidders’ equilibrium bid distributions are the same in the first-price auction.

I don’t want to get into their formal proofs, because they require some notation. But the intuition is fairly clean.

Consider some type $t_1$ of bidder 1 whose equilibrium bid is $b$. And suppose that the type of bidder 2 who ties with him is $t_2 > t_1$. (We know that with “weak” and “strong” bidders, weak bidders bid more aggressively given the same valuation, so this would be the case if bidder 1 was weak.)

So if bidder 1 wins, he knows that there is still a chance that $t_2 > t_1$; so he’ll pick some price $p > t_1$ and offer to resell the object at that price, which will occur some of the time. Call the price that maximizes his expected gain $p(b)$.

Now fix $p(b)$ at that level, and consider what happens if bidder 1 considers raising his bid by some small amount $\epsilon$. This will have two effects: he’ll pay more when he wins, and he’ll win more often.

The additional amount he pays is $\epsilon G_2(b)$, where $G_2$ is the equilibrium distribution of bidder 2’s bids (and therefore the probability of winning at a given bid). The amount more
often that he wins is $\epsilon g_2(b)$.

Now here’s the fun part. What does he gain the times he wins? Well, we know that $t_2$ may be greater than $t_1$, so we know that sometimes, he’s reselling the object at $p(b)$. Now when he raises his bid, the types of bidder 2 that he used to lose to but now outbids become the highest types who lose to him; therefore, these are the types that will definitely accept his resale offer. That is, when 1 raises his bid slightly and wins more often, all these new cases of winning lead to resale; so each one is worth $p(b) - b$, the difference between the resale price and the price he paid. Of course, back when he was losing, he was getting 0, because since $t_2 > t_1$ in this range, resale was worthless.

So the first-order condition from bidder 1’s optimal bid problem leads to the fact that

$$-G_2(b) + g_2(b)(p(b) - b) = 0$$

or

$$\frac{g_2(b)}{G_2(b)} = \frac{1}{p(b) - b}$$

But now we can do the same analysis for bidder 2. He knows that when he bids $b$ and wins, $t_2 > t_1$, so there’s no reason to resell the object; but that when he just barely loses, he will rebuy the object. So if he thinks about raising his bid slightly, he will pay an addition $\epsilon$ each of the $G_1(b)$ times he wins; but he will also win more often, by $\epsilon g_1(b)$. And these additional times that he wins are each worth $p(b) - b$, because now he wins the auction at a price $b$, rather than losing and having to buy the object for $p(b)$ from the winner.

So once again, the first-order condition leads to

$$\frac{g_1(b)}{G_1(b)} = \frac{1}{p(b) - b}$$

so

$$\frac{g_1(b)}{G_1(b)} = \frac{g_2(b)}{G_2(b)}$$

or

$$\frac{\partial}{\partial b} \log G_1(b) = \frac{\partial}{\partial b} \log G_2(b)$$

Since equilibrium bid functions must have the same range, this establishes that $G_1 = G_2$.

So what’s going on here? We showed before that with independent types, common values led to symmetry of the bid distributions. In this case, we generally have private values. But on the margin – that is, conditioning on the two bidders’ bids being very close together – resale is sure to happen when the outcome is inefficient. And so when the bids are close together, both bidders value winning the auction the same amount – at the difference between the resale price and the winning bid. So while common values do not hold generally, they do hold at the relevant point – where the bidders’ first-order conditions are determined – and so we end up with the same symmetry.

Hafalir and Krishna consider other resale protocols besides the winner making a take-it-or-leave-it offer; they allow for the loser to make a take-it-or-leave-it buy offer; or for one
of them to be chosen at random to make the offer. However, in these cases, it becomes
important that neither the winning nor the losing bid be announced prior to the resale
stage.

They then go on to show that from the seller’s point of view, the first-price auction
revenue-dominates the second-price auction when resale is a possibility. The techniques
they use for this are pretty advanced. But it’s a pretty nice result.

Next week...

Next week, we’ll talk about information aggregation in common-value auctions, and then
move on to endogenous information, with the Persico and Larson papers (and my own).