Tuesday, we laid a bunch of technical groundwork for our analysis of auctions in the Milgrom-Weber framework with interdependent values and affiliated signals. Today, we get back to auctions.

**General Environment**

- $N$ risk-neutral bidders
- Each bidder gets signal $t_i \in \mathbb{R}$
- Some other information $t_0$ out there as well – today we’ll assume the seller has it
- Each bidder’s value for the object is
  \[ v_i(t_i, t_{-i}, t_0) = v(t_i, t^{(1)}, t^{(2)}, \ldots, t^{(N-1)}, t_0) \]
  - $v$ is nonnegative and nondecreasing in all arguments (generally strictly increasing in own signal)
  - $(t_1, t_2, \ldots, t_N, t_0)$ are affiliated, with joint density $f$ which is symmetric in its first $N$ arguments

Recall that this nests both private values and common values as special cases; and that independence is a special case of affiliation.

**The Winner’s Curse**

Once we move outside of the private values framework, we come upon a phenomenon known as the Winner’s Curse. The winner’s curse is basically this: whatever you bid, you will win the auction whenever everyone else bids less than you. But in an interdependent values setting, whatever information caused them to bid low, also reduces the value of the object to you; so conditional on winning the object, its expected value is lower than it would have been if, say, you had lost the auction. (Basically, the winner’s curse is another name for an adverse selection problem – as long as your valuation is increasing on your opponents’ signals and bid functions are increasing, the object is less valuable conditional on winning it than conditional on losing it.) This will be built into the equilibrium bid functions.
Ascending Auctions

With private values, our analysis of ascending auctions was fairly simple. When you know exactly what the object for sale is worth to you, it’s a dominant strategy not to bid higher than this value, but also not to let the auction end at a price lower than this value; and any information you pick up over the course of the auction doesn’t change this fact. Thus, we argued that an ascending auction was well-approximated by either a button auction or a sealed-bid second-price auction, and went from there.

With interdependent values, this equivalence breaks down. When your own valuation depends nontrivially on the signals of other bidders, information you gain about those signals over the course of the auction causes you to revise your estimate of the value of winning, and thus changes your strategy. In a free-form ascending auction, you get a noisy signal of how many bidders are still active – really, you have a lower bound on the number of bidders still interested at each given price, but not the exact number, since a bidder may be quiet throughout an ascending auction and then jump in with a big bid at the end. In addition, once a bidder has lost interest, you don’t know at exactly what price this happened. This makes “true” ascending auctions very difficult to analyze in an interdependent values setting.

Milgrom and Weber deal with this by analyzing two different idealized versions of an ascending auction.

Button Auction with Minimal Information

The first of these is the Button Auction with Minimal Information. In this auction, each bidder sits in a separate room with their finger on a button. The price rises continuously, and bidders stay active by holding down the button or drop out (permanently) by releasing it. The auction ends when all but one bidder has dropped out, and he pays the price at which the second-to-last bidder dropped out; and no new information is given to the bidders until the auction ends. That is, at any given time, the bidders do not know how many other bidders are still active, or at what prices they’ve dropped out.

The button auction with minimal information turns out to be equivalent to a sealed-bid second-price auction. Since you don’t receive any new information until it’s too late, all you can do in this auction is to pick the price at which you’ll give up if you haven’t already won; the bidder who picks the highest dropout price wins, and his payment is equal to the dropout price of the second-highest bidder.

Our first result is the symmetric equilibrium of the button auction with minimal information.

For ease of notation, Milgrom and Weber often focus on the bidder 1. Let \( t^{(i)} \) denote the \( i^{th} \) highest of \( \{t_2, \ldots, t_N\} \). Define

\[
\hat{v}(r, s) = E \left( v_1(\cdot) \| t_1 = r, t^{(1)} = s \right)
\]
that is, \(\hat{v}(r, s)\) is the expected value of the object to bidder 1, conditioning on \(t_1 = r\) and
the highest of his opponents’ signals being equal to \(s\). (By symmetry, \(\hat{v}\) applies the same
way to all the other bidders as well.)

**Lemma 1.** Suppose \(\hat{v}\) is strictly increasing in its first argument and weakly increasing in its
second. Then every bidder playing the strategy \(b_i(t_i) = \hat{v}(t_i, t_i)\) is a symmetric equilibrium
of the button auction with minimal information.

That is, an equilibrium of this auction is for every bidder to bid the expected value of
their valuation, conditioned on their own signal, and conditioned on the highest of their
opponents’ signals matching their own.

In fact, this is a little bit more that an equilibrium; it has the additional property
that if bidder \(i\) learned the signal (and bid) of his strongest competitor, he could not gain
by changing his bid. This is how we prove this is an equilibrium. Again, it’s simpler
notationally if we focus on bidder 1.

Suppose bidder 1 learned \(t^{(1)}\), and knew that everyone else was following this equilibrium
strategy, so that the highest bid he was up against was \(\hat{v}(t^{(1)}, t^{(1)})\). Bidder 1 expects to
value the object at exactly \(\hat{v}(t_1, t^{(1)})\). So his expected payoff from bidding \(b\) is

\[
\left(\hat{v}(t_1, t^{(1)}) - \hat{v}(t^{(1)}, t^{(1)})\right) 1_{b > \hat{v}(t^{(1)}, t^{(1)})}
\]

Of course, since \(\hat{v}\) is increasing in its first argument, it is positive when \(t_1 > t^{(1)}\)
and negative when \(t_1 < t^{(1)}\). So bidder 1 wants to outbid his strongest opponent when \(t_1 > t^{(1)}\), and not
when \(t_1 < t^{(1)}\). Of course, \(\hat{v}(s, s)\) is strictly increasing in \(s\), so bidding \(\hat{v}(t_1, t_1)\) accomplishes
exactly that. So bidding \(\hat{v}(t_1, t_1)\) is a best-response for any realization of \(t^{(1)}\); so it is also a
best-response when bidder 1 doesn’t know \(t^{(1)}\). The same argument holds for each bidder
\(i\), so this strategy is an equilibrium.

Here, we showed that \(b(t) = \hat{v}(t, t)\) is an equilibrium, and that it remains a best-response
when bidder \(i\) learns the type (and value) of the highest of his competitors. However, when
\(N > 2\), it does not remain a best-response if bidder \(i\) were to learn the values and bids of
all his competitors, since these would cause him to revise his estimate of \(v\) and therefore to
bid differently. Equilibria that are robust to learning your opponents’ types and bids are
called *ex post equilibria*. We saw earlier that with private values, bidding your type was
a dominant strategy; so everyone bidding their type is not only an equilibrium, but an ex
post equilibrium. Here, we just saw that for \(N = 2\), both bidders bidding \(\hat{v}(t, t)\) is an ex
post equilibrium; but it is not when \(N > 2\).

This equilibrium is generally the only symmetric equilibrium of this auction; but it is
not always the only equilibrium. Later on, when we look at common value auctions, we’ll
see examples where there are lots and lots of asymmetric equilibria.
Button Auction with Maximal Information

In the button auction with minimal information, the bidders learn nothing about other bidders’ behavior until the auction ends. At the other extreme is the button auction with maximal information. Here, at every price, the bidders know exactly who is still active, and the price at which the inactive bidders dropped out. Since equilibrium strategies will be strictly increasing, this allows bidders to infer the exact type of every bidder who has already dropped out.

There turns out to be a symmetric ex post equilibrium for the button auction with maximal information. The notation is a bit messy, but the intuition is fairly clean. Basically, at each point, bidder \( i \) knows the type of the bidders who are inactive; he computes the value of \( v_i \), conditioned on his own type, the types of the bidders who have dropped out, and assuming that all remaining bidders have types which exactly match his own. You stay in until this value is reached, and drop out when it is. Each time another bidder drops out, you infer their type, and redo this calculation.

Formally, let \( \beta_0(t) \) denote bidder \( i \)'s strategy (dropout point) at type \( t \) if nobody else has dropped out yet, and \( \beta_n(p_1, \ldots, p_n) \) bidder \( i \)'s strategy when \( n \) bidders have already dropped out, at prices \( p_1, p_2, \ldots, p_n \). For ease of notation, recall that \( v \) is the primitive value function, and depends on \( t_1, \ldots, t_n, t_0 \); for a given type profile, let \( \bar{v}(t_1, \ldots, t_n) \) be the expected value of \( v \) at that type profile (the expectation is taken over \( t_0 \), conditioning on the types).

**Lemma 2.** All bidders playing the following strategies constitute a symmetric ex post equilibrium of the button auction with maximal information:

\[
\begin{align*}
\beta_0(t) &= \bar{v}(t, t, \ldots, t) \\
\beta_n(t, p_1, \ldots, p_n) &= \bar{v}

\text{where } \hat{t}^{(N-k)} \text{ are recursively defined as the solutions to}

\begin{align*}
p_k &= \beta_{k-1}(\hat{t}^{(N-k)}, p_1, \ldots, p_{k-1}) \\
&= \bar{v}(\hat{t}^{(N-k)}, \ldots, \hat{t}^{(N-k)}, \hat{t}^{(N-(k-1))}, \ldots, \hat{t}^{(N-1)})
\end{align*}
\]

To prove this, first note that at any stage, when a new bidder drops out, it’s never enough to make you wish you had dropped out earlier. So there are no “domino effects” where a bunch of people drop out at the same time. Since \( \bar{v} \) is continuous and strictly increasing in its first argument, dropout prices uniquely identify the types of the bidders who drop out.

To see this is an ex post equilibrium, note that if everyone else plays this strategy and bidder 1 wins, he pays

\[
\bar{v}

\text{and values winning at}

\bar{v}
\]
So he wants to win whenever $t_1 > t^{(1)}$, and wants to lose whenever $t_1 < t^{(1)}$, which he accomplishes by playing this strategy. Since this is an ex post equilibrium, it’s an equilibrium.

It’s been pointed out that there are other symmetric equilibria, but they give the same result (the same bidder wins and pays the same amount).

Also, note that in both these auctions, the bidder with the highest type wins. However, we haven’t made any assumptions that this is the bidder with the highest value. This is true as long as the valuation function $v$ is more sensitive to your own signal than to another bidder’s signal.

**Lemma 3.** Suppose that for $t > t'$, $v(t, t, t_{-ij}) \geq v(t', t, t_{-ij})$. Then the equilibria above for both types of button auctions are efficient.

**Revenue Comparisons**

Here’s where it begins to get fun.

**Theorem 1.** For any realization of the highest two bidders’ signals, the expected price paid by the winner is at least as great in the maximal-information button auction as in the minimal-information button auction.

Of course, we can then take the expectation over the joint distribution of the highest two types to see that expected revenue is higher in the max-information than in the min-information button auction.

Let’s look at the case where bidder 1 has the highest type, and bidder 2 the second-highest, and let $x = t_1$ and $y = t_2$. The minimum-information button auction is just a second-price auction, so bidder 1 pays bidder 2’s bid, $\hat{v}(y,y)$, which is equal to (by symmetry)

$$E_{t_3,\ldots,t_N|t_1=x,t_2=y,t_3,\ldots,t_N<y}v_2(y,y,t_3,\ldots,t_N)$$

where the distribution of $t_3$ through $t_N$ is conditioned on $t_1 = t_2 = y$. On the other hand, in the maximum-information button auction, $t_3$ through $t_N$ are revealed through equilibrium bidding, and so bidder 1 pays the price at which bidder 2 drops out, which is

$$v_2(t_1 = y, t_2 = y, t_3, t_4, \ldots, t_N)$$

(although we still should take the expected value over $t_0$). This latter one is based on the actual values of $t_3$ through $t_N$; so if we fix $t_1$ and $t_2$ at $x$ and $y$ and take the expectation, we get

$$E_{t_3,\ldots,t_N|t_1=x,t_2=y,t_3,\ldots,t_N<y}v_2(y,y,t_3,\ldots,t_N)$$

where now the distribution of $t_3$ through $t_N$ is conditioned on $t_1 = x$ and $t_2 = y$. Since $x > y$, the variables are affiliated, and $v_2$ is increasing, this latter expectation is at least as high.

Similar reasoning leads to the following result:
Lemma 4. In either type of button auction, a policy of always publicizing the realization of \( t_0 \) leads to weakly higher expected revenue than a policy of always keeping \( t_0 \) secret.

Again, this ends up being true for each realization of the highest two bidders’ types, and therefore true overall. We will show this for the minimal-information case; the proof for the maximal-information case is nearly identical.

Again suppose that \( t_1 > t_2 > \max\{t_3, \ldots, t_N\} \), and let \( x = t_1 \) and \( y = t_2 \). If \( t_0 \) is revealed, revenue is equal to bidder 2’s bid, which is

\[
E_{t_3, \ldots, t_N|t_1 = t_2 = y, t_0} (v_2(y, y, t_3, \ldots, t_N, t_0))
\]

where \( t_0 \) is the actual realization of the seller’s information, and the distribution of \( t_3, \ldots, t_N \) is conditioned on \( t_1 = t_2 = y \) and \( t_0 \). Taking the expectation over \( t_0 \), conditioned on the actual realizations of \( t_1 \) and \( t_2 \), gives

\[
E_{t_0|t_1 = x, t_2 = y} \left\{ E_{t_3, \ldots, t_N|t_1 = t_2 = y, t_0} (v_2(y, y, t_3, \ldots, t_N, t_0)) \right\}
\]

On the other hand, if \( t_0 \) is not revealed, revenue is

\[
E_{t_3, \ldots, t_N, t_0|t_1 = y, t_2 = y} (v_2(y, y, t_3, \ldots, t_N, t_0))
\]

which can be rewritten using iterated expectations as

\[
E_{t_0|t_1 = y, t_2 = y} \left\{ E_{t_3, \ldots, t_N|t_1 = y, t_2 = y, t_0} (v_2(y, y, t_3, \ldots, t_N, t_0)) \right\}
\]

The two expressions are the same, except that in the first, the distribution of \( t_0 \) is conditioned on the truth \((t_1 = x, t_2 = y)\), and in the second, the distribution of \( t_0 \) is conditioned on the assumption that bidder 2 uses to calculate his bid \((t_1 = y, t_2 = y)\). But the expression within curly brackets is increasing in \( t_0 \) (since \((t_3, \ldots, t_N, t_0)\) are affiliated and \( v_2 \) is increasing in all its arguments); and since \((t_0, t_1, t_2)\) are affiliated, conditioning on \( t_1 = x \) instead of \( t_1 = y \) gives a higher (by FOSD) distribution of \( t_0 \), leading to a higher result in the first than in the second.

The proof for the maximum-information case is nearly the same.

In fact, Milgrom and Weber prove something a little stronger: not only is always revealing \( t_0 \) better than never revealing \( t_0 \), but it’s better than any other reporting policy, such as reporting \( t_0 \) plus noise, only revealing \( t_0 \) when it’s within a certain range, etc.
First Price Auctions

Now we consider the sealed-bid first-price auction in the same setting. Recall that

\[ \hat{v}(x, y) = E(v_1|t_1 = x, t^{(1)} = y) \]

We also define \( F(t^{(1)}|t_1) \) as the conditional cumulative distribution function of \( t^{(1)} \), the highest opponent type, conditional on a realization \( t_1 \) of my own type, and \( f(t^{(1)}|t_1) \) as the corresponding density.

**Lemma 5.** The following is a symmetric equilibrium strategy for the first-price auction:

\[ \beta(s) = \hat{v}(0, 0) + \int_0^s \hat{v}(\alpha, \alpha) dL(\alpha|s) \]

where

\[ L(\alpha|s) = \exp\left(-\int_\alpha^s \frac{f(z|s)}{F(z|s)} dz\right) \]

Given true value of \( t_1 = s \) and a bid of \( x \), we can write bidder 1’s expected payoff as

\[ E\left( (\hat{v}(s, t^{(1)}) - x) 1_{x > \beta(t^{(1)})}|t_1 = s \right) \]

or as

\[ \int_0^{\beta^{-1}(x)} (\hat{v}(s, \tau) - x) f(\tau|s) d\tau \]

Differentiating with respect to \( x \) gives

\[ -\int_0^{\beta^{-1}(x)} f(\tau|s) d\tau + (\beta^{-1})'(x) (\hat{v}(s, \beta^{-1}(x)) - x) f(\beta^{-1}(x)|s) \]

Plugging in \( x = \beta(s) \) gives

\[ -F(s|s) + \frac{1}{\beta(s)} (\hat{v}(s, s) - \beta(s)) f(s|s) \]

Setting this equal to 0 gives

\[ \beta'(s) = (\hat{v}(s, s) - \beta(s)) \frac{f(s|s)}{F(s|s)} \]

The proof that the strategies above are an equilibrium involves showing that they solve this differential equation, with the correct boundary condition; which means they satisfy the envelope condition (modified slightly to account for interdependence of types); which, along with the fact that \( \beta \) is increasing, turns out to be sufficient for it to be a best-response, and therefore an equilibrium.

Where this gets us is:

**Theorem 2.** For each type of bidder, the expected price paid in the second-price auction, conditional on winning, is at least as high as the bid in the first-price auction.
Recall that the equilibrium bid in the second-price auction is $\hat{v}(t, t)$, so this says that

$$E\left(\hat{v}(t^{(1)}, t^{(1)})|t^{(1)} < t_1 = s\right) \geq \beta(s)$$

Let $\beta^F$ denote the equilibrium bid function in the first-price auction, and define

$$\beta^S(s, t) = E\left(\hat{v}(t^{(1)}, t^{(1)})|t^1 = s, t^{(1)} \leq t\right)$$

as the expected price that bidder 1 expects to pay in the second-price auction, conditional on having a true type of $s$ but bidding as if he were type $t$. We want to show that $\beta^F(s) \leq \beta^S(s, s)$.

Let

$$\tilde{v}(s, t) = E\left(\hat{v}(s, t^{(1)})|t^1 = s, t^{(1)} < t\right)$$

be the expected value of the object to bidder 1, conditional on bidder 1 having true type $s$, bidding as if he were type $t$, and winning. Then his expected payoff is

$$V^F(s) = \max_t \left(\tilde{v}(s, t) - \beta^F(t)\right) F(t|s)$$

where $F$ is once again the cumulative distribution function of his highest opponent’s type. In equilibrium, this max is achieved by setting $t = s$, so by the envelope theorem, $V'$ is the $s$-derivative of the right-hand side, evaluated at $t = s$, which is (using numbers to denote partial derivatives)

$$(V^F)'(s) = \tilde{v}_1(s, s) F(s|s) + (\tilde{v}(s, s) - \beta^F(s)) F_2(s|s)$$

In the second-price auction,

$$V^S(s) = \max_t \left(\tilde{v}(s, t) - \beta^S(s, t)\right) F(t|s)$$

and so

$$(V^S)'(s) = \tilde{v}_1(s, s) F(s|s) - \beta^S(s, s) F(s|s) + (\tilde{v}(s, s) - \beta^S(s, s)) F_2(s|s)$$

Now, a slight variation on the ranking lemma we already used:

**Lemma 6.** Suppose $g, h$ are continuous differentiable functions, and $g(0) \geq h(0)$. If for all $x \geq 0$, $g(x) \leq h(x)$ implies $g'(x) \geq h'(x)$, then $g(x) \geq h(x)$ for all $x$.

Now, $V^F(s) = (\tilde{v}(s, s) - \beta^F(s)) F(s|s)$ and $V^S(s) = (\tilde{v}(s, s) - \beta^S(s, s)) F(s|s)$, so $V^F < V^S$ if and only if $\beta^F(s) > \beta^S(s, s)$. We will show this is impossible.

We know that $V^F(0) = V^S(0) = 0$. If $V^F(s) \leq V^S(s)$, then $\beta^F(s) \geq \beta^S(s, s)$. But we know (we proved last time) that $F(t|s)$ is decreasing in $s$, so $F_2(s|s)$ is negative. And by affiliation, $\beta^S$ is positive. Comparing the envelope theorem equations gives $V^F(s) \leq V^S(s) \rightarrow (V^F)'(s) \geq (V^S)'(s)$, so by the ranking lemma, $V^F(s) \geq V^S(s)$ for all $s$; so $\beta^F(s) \leq \beta^S(s, s)$.

We also have the same result as before:

**Theorem 3.** For each type of bidder, the expected payment by that bidder, conditional on winning, when the seller reveals $t_0$ is at least as high as the expected payment when the seller does not reveal $t_0$. 

8
The Linkage Principle

We’ve now shown that from an expected revenue point of view,

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(Start with bottom row, then “similarly show” top row.)

It’s also clear that, as you move up and to the left on this chart, more information is revealed, and in particular, the winning bidder’s payment (in equilibrium) is a function of more information.

This is an illustration of the linkage principle:

“The more closely the winning bidder’s payment is linked to his actual type (as opposed to his bid), the greater the expected revenue will be”

which, in this case, can be rephrased as,

“The more things the winning bidder’s payment depends on that are positively correlated with his type, the greater the expected revenue will be”

(One way to think about this is that bidder’s expected payoff can be thought of as his “information rents,” that is, the extra surplus he is able to get by having private information. But in auction formats where information is revealed which is correlated with his private information, his private information becomes “less private” in a sense, so he gets a smaller surplus, and therefore more goes to the seller. In the logical limit – where information revealed over the course of the auction fully reveals the highest type – the seller could simply make a take-it-or-leave-it offer to extract full surplus.)

Other Stuff

Milgrom and Weber also give some results on auctions with reserve prices, and auctions with reserve prices and entry fees; as well as some results on risk-averse bidders (but only with constant absolute risk aversion).

We found two weeks ago that risk aversion favored the first-price auction over the ascending auction; we found today that affiliation of types favored the ascending auction over the first-price auction. The question of what to do when there is both risk aversion and affiliation of signals is tricky; Paul Klemperer has one clever solution, which we’ll get into next week (along with an example fleshed out by Dan Levin and Lixin Ye in a note I just added to the syllabus).