1. Define

\[ g(x, \beta) = 1(x \leq \beta) - \frac{1}{2} \]

where \( 1(\cdot) \) is the indicator function (takes the value 1 if the argument is true, else equals zero).

Let \( \beta \) satisfy \( E g(X_i, \beta) = 0 \). Is \( \beta \) the median of the distribution of \( X_i \)?

2. Let \( \hat{\beta} \) satisfy \( \overline{g}_n(\hat{\beta}) = 0 \) where \( \overline{g}_n(b) = n^{-1} \sum_{i=1}^n g(X_i, b) \) and \( g \) is defined as in problem 1. Show that \( \hat{\beta} \) is the sample median.

3. Let \( X \) be a random variable with \( \mu = EX \) and \( \sigma^2 = Var(X) \). Define

\[ g(x, \mu, \sigma^2) = \left( \frac{x - \mu}{(x - \mu)^2 - \sigma^2} \right). \]

Let \( (\hat{\mu}, \hat{\sigma}^2) \) be the values such that \( \overline{g}_n(\hat{\mu}, \hat{\sigma}^2) = 0 \) where \( \overline{g}_n(m, s) = n^{-1} \sum_{i=1}^n g(X_i, m, s) \). Show that \( \hat{\mu} \) and \( \hat{\sigma}^2 \) are the sample mean and variance.

4. Take the bi-variate linear projection model

\[ y_i = \beta_0 + \beta_1 x_i + e_i \]

\[ Ee_i = 0 \]

\[ Ex_ie_i = 0 \]

Define \( \mu_y = E y_i, \mu_x = E x_i, \sigma^2_x = Var(x_i), \sigma^2_y = Var(y_i) \) and \( \sigma_{xy} = Cov(x_i, y_i) \). Show that \( \beta_1 = \sigma_{xy}/\sigma^2_x \) and \( \beta_0 = \mu_y - \beta_1 \mu_x \).

5. Suppose that \( y_i \) is discrete-valued, taking values only on the non-negative integers, and the conditional distribution of \( y_i \) given \( x_i \) is Poisson:

\[ P(y_i = k \mid x_i = x) = \frac{e^{-x}(x^k)}{k!}, \quad k = 0, 1, 2, \ldots \]

Compute \( E(y_i \mid x_i = x) \) and \( Var(y_i \mid x_i = x) \). Does this justify a linear regression model of the form \( y_i = x_i^T \beta + \varepsilon_i \)?

Hint: If \( P(Y = k) = \frac{e^{-\lambda} \lambda^k}{k!} \), then \( EY = \lambda \) and \( Var(Y) = k \).

6. Let \( x_i \) and \( y_i \) have the joint density \( f(x, y) = \frac{3}{2} (x^2 + y^2) \) on \( 0 \leq x \leq 1, 0 \leq y \leq 1 \). Compute the coefficients of the linear projection \( y_i = \beta_0 + \beta_1 x_i + e_i \). Compute the conditional mean \( m(x) = E(y_i \mid x_i = x) \). Are they different?