1. In the linear model $Y = X\beta + e$ with $E(x_ie_i) = 0$, the Generalized Method of Moments (GMM) criterion function for $\beta$ is defined as

$$J_n(\beta) = \frac{1}{n} \left( Y - X\beta \right)' X \hat{\Omega}_n^{-1} X \left( Y - X\beta \right)$$

(1)

where $\hat{e}_i$ are the OLS residuals and $\hat{\Omega}_n = \frac{1}{n} \sum_{i=1}^{n} x_i x_i' \hat{e}_i^2$. The GMM estimator of $\beta$, subject to the restriction $h(\beta) = 0$, is defined as

$$\tilde{\beta} = \text{arg min}_{h(\beta) = 0} J_n(\beta).$$

The GMM test statistic (the distance statistic) of the hypothesis $h(\beta) = 0$ is

$$D = J_n(\tilde{\beta}) = \min_{h(\beta) = 0} J_n(\beta).$$

(2)

(a) Show that you can rewrite $J_n(\beta)$ in (1) as

$$J_n(\beta) = \left( \beta - \hat{\beta} \right)' \hat{\Omega}_n^{-1} \left( \beta - \hat{\beta} \right)$$

where

$$\hat{\Omega}_n = \left( X'X \right)^{-1} \left( \sum_{i=1}^{n} x_i x_i' \hat{e}_i^2 \right) \left( X'X \right)^{-1}. $$

(b) Now focus on linear restrictions: $h(\beta) = R'\beta - r$. Thus

$$\tilde{\beta} = \text{arg min}_{R'\beta - r} J_n(\beta)$$

and hence $R'\tilde{\beta} = r$. Define the Lagrangian $L(\beta, \lambda) = \frac{1}{2} J_n(\beta) + \lambda' \left( R'\beta - r \right)$ where $\lambda$ is $s \times 1$. Show that the minimizer is

$$\tilde{\beta} = \hat{\beta} - \hat{V}_nR \left( R'\hat{V}_nR \right)^{-1} \left( R'\tilde{\beta} - r \right)$$

(3)

$$\hat{\lambda} = \left( R'\hat{V}_nR \right)^{-1} \left( R'\tilde{\beta} - r \right).$$

(c) Show that if $R'\beta = r$ then $\sqrt{n} \left( \tilde{\beta} - \beta \right) \rightarrow_d N(0, V_R)$ where

$$V_R = V - VR \left( R'VR \right)^{-1} R'V. $$

(4)

(d) Show that in this setting, the distance statistic $D$ in (2) equals the Wald statistic.
2. Take the linear model

\[ y_i = z_i' \beta + e_i \]
\[ E(x_i e_i) = 0. \]

and consider the GMM estimator \( \hat{\beta} \) of \( \beta \). Let

\[ J_n = n \overline{g}_n(\hat{\beta}')(\hat{\Omega}^{-1}) \overline{g}_n(\hat{\beta}) \]

denote the test of overidentifying restrictions. Define

\[
D_n = I_l - C' \left( \frac{1}{n} X' Z \right) \left( \frac{1}{n} Z' X \hat{\Omega}^{-1} \frac{1}{n} X' Z \right)^{-1} \frac{1}{n} Z' X \hat{\Omega}^{-1} C'^{-1} \\
\overline{g}_n(\beta_0) = \frac{1}{n} X'e \\
R = C'E (x_i z_i')
\]

Show that \( J_n \to_d \chi^2_{l-k} \) as \( n \to \infty \) by demonstrating each of the following:

(a) Since \( \Omega > 0 \), we can write \( \Omega^{-1} = CC' \) and \( \Omega = C'^{-1}C^{-1} \)

(b) \( J_n = n \left(C' \overline{g}_n(\hat{\beta})\right)' \left(C' \hat{\Omega} C\right)^{-1} C' \overline{g}_n(\hat{\beta}) \)

(c) \( C' \overline{g}_n(\beta) = D_n C' \overline{g}_n(\beta_0) \)

(d) \( D_n \to_p I_l - R (R'R)^{-1} R' \)

(e) \( n^{1/2} C' \overline{g}_n(\beta_0) \to_d N \sim N (0, I_l) \)

(f) \( J_n \to_d N' \left( I_l - R (R'R)^{-1} R' \right) N \)

(g) \( N' \left( I_l - R (R'R)^{-1} R' \right) N \sim \chi^2_{l-k} \).

Hint: \( I_l - R (R'R)^{-1} R' \) is a projection matrix.

3. Consider the single equation model

\[ y_i = z_i \beta + e_i, \]

where \( y_i \) and \( z_i \) are both real-valued \( (1 \times 1) \). Let \( \hat{\beta} \) denote the IV estimator of \( \beta \) using as an instrument a dummy variable \( d_i \) (takes only the values 0 and 1). Find a simple expression for the IV estimator in this context.

4. In the linear model

\[ y_i = z_i' \beta + e_i \]
\[ E(e_i \mid z_i) = 0 \]

suppose \( \sigma^2_i = E(e_i^2 \mid z_i) \) is known. Show that the GLS estimator of \( \beta \) can be written as an IV estimator using some instrument \( x_i \). (Find an expression for \( x_i \).)