1. Take the single equation

\[ Y = Z\beta + e \]

\[ E(e | X) = 0 \]

Assume \( E(e_i^2 | x_i) = \sigma^2 \). Show that if \( \hat{\beta} \) is estimated by GMM with weight matrix \( W_n = (X'X)^{-1} \), then

\[ \sqrt{n} \left( \hat{\beta} - \beta \right) \rightarrow_d N(0, \sigma^2 (Q'M^{-1}Q)^{-1}) \]

where \( Q = E(x_i z_i') \) and \( M = E(x_i x_i') \).

2. Take the model \( y_i = z_i'\beta + e_i \) with \( E(x_i e_i) = 0 \). Let \( \hat{e}_i = y_i - z_i'\hat{\beta} \) where \( \hat{\beta} \) is consistent for \( \beta \) (e.g. a GMM estimator with arbitrary weight matrix). Define the estimate of the optimal GMM weight matrix

\[ W_n = \left( \frac{1}{n} \sum_{i=1}^{n} x_i x_i' \hat{e}_i^2 \right)^{-1}. \]

Show that \( W_n \rightarrow_p \Omega^{-1} \) where \( \Omega = E(x_i x_i' e_i^2) \).

3. In the linear model estimated by GMM with general weight matrix \( W \), the asymptotic variance of \( \hat{\beta}_{GMM} \) is

\[ V = (Q'WQ)^{-1} Q'W\Omega WQ (Q'WQ)^{-1} \]

(a) Let \( V_0 \) be this matrix when \( W = \Omega^{-1} \). Show that \( V_0 = (Q'\Omega^{-1}Q)^{-1} \).

(b) We want to show that for any \( W \), \( V - V_0 \) is positive semi-definite (for then \( V_0 \) is the smaller possible covariance matrix and \( W = \Omega^{-1} \) is the efficient weight matrix). From matrix algebra, we know that \( V - V_0 \) is positive semi-definite if and only if

\[ V_0^{-1} - V^{-1} = A \]

is positive semi-definite. Write out the matrix \( A \).

(c) Since \( \Omega \) is positive definite, there exists a nonsingular matrix \( C \) such that \( C'C = \Omega^{-1} \). Letting \( H = CQ \) and \( G = C'^{-1}WQ \), verify that \( A \) can be written as

\[ A = H' \left( I - G (G'G)^{-1} G' \right) H. \]
(d) Show that $A$ is positive semidefinite.

Hint: The matrix $I - G(G'G)^{-1}G'$ is symmetric and idempotent, and therefore positive semidefinite.

4. The equation of interest is

$$y_i = g(x_i, \beta) + e_i$$

$$E(z_i e_i) = 0.$$

The observed data is $(y_i, x_i, z_i)$. $z_i$ is $l \times 1$ and $\beta$ is $k \times 1$, $l \geq k$. Show how to construct an efficient GMM estimator for $\beta$. 