1. Take the linear model with restrictions

\[ y_i = x_i' \beta + e_i \]
\[ E(x_i e_i) = 0 \]
\[ R' \beta = c \]

with \( n \) observations. Consider three estimators for \( \beta \)

- \( \hat{\beta} \), the unconstrained least squares estimator
- \( \bar{\beta} \), the constrained least squares estimator
- \( \overline{\beta} \), the constrained efficient minimum distance estimator

For each estimator, define the residuals \( \hat{e}_i = y_i - x_i' \hat{\beta} \)
\( \bar{e}_i = y_i - x_i' \bar{\beta} \), \( v_i = y_i - x_i' \overline{\beta} \), and variance estimators \( \hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} e_i^2 \)
\( \bar{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} \bar{e}_i^2 \), and \( \overline{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} v_i^2 \).

(a) As \( \overline{\beta} \) is the most efficient and \( \hat{\beta} \) the least, do you expect that \( \overline{\sigma}^2 < \bar{\sigma}^2 < \hat{\sigma}^2 \), in large samples?
(b) Consider the statistic
\[ T_n = \frac{1}{n} \sum_{i=1}^{n} (\hat{e}_i - \bar{e}_i)^2 \]
Find the asymptotic distribution for \( T_n \) when \( R' \beta = c \) is true.
(c) Does the result of the previous question simplify when the error \( e_i \) is homoskedastic?

2. Take the linear model

\[ y_i = x_{1i} \beta_1 + x_{2i} \beta_2 + e_i \]
\[ E(x_i e_i) = 0 \]

with \( n \) observations. Consider the restriction

\[ \frac{\beta_1}{\beta_2} = 2 \]  \hspace{1cm} (1)

(a) Find an explicit expression for the constrained least-squares (CLS) estimator \( \bar{\beta} = (\beta_1, \beta_2) \) of \( \beta = (\beta_1, \beta_2) \) under (1). Your answer should be specific to the restriction (1), it should not be a generic formula for an abstract general restriction.
(b) Derive the asymptotic distribution of \( \beta_1 \) under the assumption that (1) is a true restriction

3. Suppose that for a pair of observables \((y_i, x_i)\) with \( x_i > 0 \) that an economic model implies

\[ E(y_i \mid x_i) = (\gamma + \theta x_i)^{1/2}. \]  \hspace{1cm} (2)

A friend suggests that (given an iid sample) you estimate \( \gamma \) and \( \theta \) by the linear regression of \( y_i^2 \) on \( x_i \), that is, to estimate the equation

\[ y_i^2 = \alpha + \beta x_i + e_i. \]  \hspace{1cm} (3)

(a) Investigate your friend’s suggestion. Define \( u_i = y_i - (\gamma + \theta x_i)^{1/2} \). Show that \( E(u_i \mid x_i) = 0 \) is implied by (2).
(b) Use \( y_i = (\gamma + \theta x_i)^{1/2} + u_i \) to calculate \( E(y_i^2 \mid x_i) \). What does this tell you about the implied equation (3)?
(c) Can you recover either \( \gamma \) and/or \( \theta \) from estimation of (3)? Are additional assumptions required?
(d) Is this a reasonable suggestion?