1. The coefficient $\gamma_2$ is the best linear predictor for $e_i^2$ given $x_i$. The coefficient $\gamma_1$ is the best linear approximation to $\sigma^2(x_i)$. They are the same. To see this explicitly, the FOC for $\gamma_1$ is

$$0 = -2E (x_i \sigma^2(x_i)) + E(x_i x'_i) \gamma_1$$

so that

$$\gamma_1 = E(x_i x'_i)^{-1} E(x_i \sigma^2(x_i)).$$

The FOC for $\gamma_2$ is

$$0 = -2E (x_i e_i^2) + E(x_i x'_i) \gamma_2$$

so that

$$\gamma_2 = E(x_i x'_i)^{-1} E(x_i e_i^2)$$

By conditioning and the LIE

$$E(x_i e_i^2) = E(E(x_i e_i^2 | x_i)) = E(x_i E(e_i^2 | x_i)) = E(x_i \sigma^2(x_i))$$

and thus

$$\gamma_2 = E(x_i x'_i)^{-1} E(x_i e_i^2) = E(x_i x'_i)^{-1} E(x_i \sigma^2(x_i)) = \gamma_1$$

2. (a) From the analysis of omitted variable bias, we know that $\gamma_1 = \beta_1$ under one of two conditions:

i. $\beta_2 = 0$ in the long regression

ii. $E(x_i x^2_i) = 0$ or equivalently $E(x^3_i) = 0$

If $E(x_i) = 0$, this is equivalent to $x_i$ having zero skewness

(b) From the same argument, $\gamma_1 = \theta_1$ under one of two conditions:

i. $\theta_2 = 0$ in the long regression

ii. $E(x_i x^3_i) = 0$ or equivalently $E(x^4_i) = 0$. This is impossible. Thus the conditions for $\gamma_1 = \theta_1$ and $\gamma_1 = \beta_1$ are not similar.

3. Substituting $y_i = x_i \beta + e_i$,

$$\hat{\beta} = \frac{\sum^n_{i=1} x^3_i y_i}{\sum^n_{i=1} x^4_i} = \frac{\sum^n_{i=1} x^3_i (x_i \beta + e_i)}{\sum^n_{i=1} x^4_i} = \beta + \frac{\sum^n_{i=1} x^3_i e_i}{\sum^n_{i=1} x^4_i}$$

Thus

$$\sqrt{n} (\hat{\beta} - \beta) = \frac{1}{\sqrt{n}} \frac{\sum^n_{i=1} x^3_i e_i}{\frac{1}{n} \sum^n_{i=1} x^4_i}$$
By the WLLN, if $Ex_i^4 < \infty$, then as $n \to \infty$

$$\frac{1}{n} \sum_{i=1}^{n} x_i^4 \to_p Ex_i^4.$$ 

By the LIE and $E(e_i \mid x_i) = 0$ then

$$E(x_i^3 e_i) = E(E(x_i^3 e_i \mid x_i)) = E(x_i^3 E(e_i \mid x_i)) = 0.$$ 

Then by the CLT, if $E(x_i^6 e_i^2) < \infty$, as $n \to \infty$,

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} x_i^3 e_i \to_d N(0, E(x_i^6 e_i^2)).$$

Together

$$\sqrt{n}(\hat{\beta} - \beta) \to_d \frac{N(0, E(x_i^6 e_i^2))}{Ex_i^4} = N(0, \frac{E(x_i^6 e_i^2)}{(Ex_i^4)^2}).$$

4.

(a) The restricted model is $y_i = \alpha + e_i$. The CLS estimator is $\hat{\alpha} = n^{-1} \sum_{i=1}^{n} y_i$.

(b) Let $\hat{\alpha}, \hat{\beta}$ be the unrestricted OLS estimator of $(\hat{\alpha}, \hat{\beta})$. Let $\hat{\mathbf{V}} = \hat{\mathbf{Q}}^{-1} \hat{\mathbf{Q}}^{-1}$ be estimator of the asymptotic covariance matrix for $(\hat{\alpha}, \hat{\beta})$. Letting $\hat{\mathbf{x}}_i = \begin{pmatrix} 1 \\ x_i \end{pmatrix}$, this is

$$\hat{\mathbf{Q}} = \frac{1}{n} \sum_{i=1}^{n} \hat{\mathbf{x}}_i \hat{\mathbf{x}}'_i$$

$$\hat{\mathbf{Q}} = \frac{1}{n} \sum_{i=1}^{n} \hat{\mathbf{x}}_i \hat{\mathbf{x}}'_i$$

where $\hat{e}_i = y_i - \hat{\alpha} - x'_i \hat{\beta}$. Partition $\hat{\mathbf{V}}$ as

$$\hat{\mathbf{V}} = \begin{bmatrix} \hat{V}_{11} & \hat{V}_{12} \\ \hat{V}_{21} & \hat{V}_{22} \end{bmatrix}.$$

Using equation (7.22) from the notes,

$$\hat{\alpha}_{MD} = \hat{\alpha} - \hat{\beta}_2 \hat{\mathbf{V}}_{22}^{-1} \hat{\beta}.$$ 

Since

$$\hat{\alpha} = \bar{y} - \bar{x} \hat{\beta}$$

we can also write this as

$$\hat{\alpha}_{MD} = \bar{y} - \left( \bar{x} + \hat{\mathbf{V}}_{22}^{-1} \hat{\mathbf{V}}_{21} \right)' \hat{\beta}.$$