1. Take the model

\[ y_i = x_i' \beta + e_i \]

\[ \mathbb{E}(z_i e_i) = 0 \]

and consider the two-stage least-squares estimator. The first-stage estimate is

\[ \hat{X} = Z \hat{\Gamma} \]

\[ \hat{\Gamma} = (Z'Z)^{-1} Z'X \]

and the second-stage is LS of \( y_i \) on \( \hat{x}_i \):

\[ \hat{\beta} = \left( \hat{X}' \hat{X} \right)^{-1} \hat{X}'Y \]

with LS residuals

\[ \hat{e} = Y - \hat{X} \hat{\beta}. \]

Consider \( \hat{\sigma}^2 = \frac{1}{n} \hat{e}' \hat{e} \) as an estimator for \( \sigma^2 = \mathbb{E}e_i^2 \). Is this appropriate? If not, propose an alternative estimator.

2. You have two independent iid samples \((y_{1i}, x_{1i}, z_{1i} : i = 1, \ldots, n)\) and \((y_{2i}, x_{2i}, z_{2i} : i = 1, \ldots, n)\). The dependent variables \(y_{1i}\) and \(y_{2i}\) are real-valued. The regressors \(x_{1i}\) and \(x_{2i}\) and instruments \(z_{1i}\) and \(z_{2i}\) are \(k\)-vectors. The model is standard just-identified linear instrumental variables

\[ y_{1i} = x_{1i}' \beta_1 + e_{1i} \]

\[ \mathbb{E}(z_{1i} e_{1i}) = 0 \]

\[ y_{2i} = x_{2i}' \beta_2 + e_{2i} \]

\[ \mathbb{E}(z_{2i} e_{2i}) = 0 \]

For concreteness, sample 1 are women and sample 2 are men. You want to test \( H_0 : \beta_1 = \beta_2 \), that the two samples have the same coefficients.

(a) Develop a test statistic for \( H_0 \)

(b) Derive the asymptotic distribution of the test

(c) Describe (in brief) the testing procedure
3. Take the model

\[ y_i = x_{1i}\beta_1 + x_{2i}\beta_2 + e_i \]

\[ \mathbb{E}(x_ie_i) = 0 \]

with both \( \beta_1 \in \mathbb{R} \) and \( \beta_2 \in \mathbb{R} \), and define the parameter \( \theta = \beta_1\beta_2 \)

(a) What is the appropriate estimator \( \hat{\theta} \) for \( \theta \)?
(b) Find the asymptotic distribution of \( \hat{\theta} \) under standard regularity conditions.
(c) Show how to calculate an asymptotic 95% confidence interval for \( \theta \)
(d) Describe how to use the percentile bootstrap to calculate a 95% confidence interval for \( \theta \)

4. You have a friend who wants to estimate \( \beta \) in the model

\[ y_i = x_i\beta + e_i \]

\[ \mathbb{E}(e_i \mid z_i) = 0 \]

with both \( x_i \in \mathbb{R} \) and \( z_i \in \mathbb{R} \), and \( z_i \) is continuously distributed. Your friend wants to treat the reduced form equation for \( x_i \) as nonparametric

\[ x_i = g(z_i) + u_i \]

\[ \mathbb{E}(u_i \mid z_i) = 0 \]

Your friend asks you for advice and help to construct an estimator \( \hat{\beta} \) of \( \beta \). Describe an appropriate estimator. You do not have to develop the distribution theory, but try to be sufficiently complete with your advice so your friend can compute \( \hat{\beta} \).