ECON 714
Assignment 4

1. The Prospect of Upward Mobility: One application of transition matrices lies in the study of income processes. It is well known that in many societies, even relatively poor people oppose high rates of redistribution in anticipation (either for them or their children) of moving up the income ladder. (For instance, in the United States, even poor people oppose the estate tax (tax on bequests), which is a tax levied on every dollar that you pass on to your kids in excess of a million dollars!) This is referred to as the prospect of upward mobility (POUM).

Imagine that income of individual $i$ at time $t$, $y_i^t$ can take one of three values $a_1$, $a_2$, and $a_3$, with $a_1 < a_2 < a_3$.

$$
\Pi = \begin{bmatrix}
1 - r & r & 0 \\
ps & 1 - s & (1 - p)s \\
0 & q & 1 - q
\end{bmatrix}.
$$

where $(p, q, r, s) \in (0, 1)$. Let the invariant distribution induced by $\Pi$ be denoted by $(\pi_1, \pi_2, \pi_3)$. We are in the business of seeking conditions on $\Pi$ such that the majority oppose redistribution. When will that be the case? Well, the majority will oppose (future) redistribution if over half of the population is always poorer than average (positively skewed income distribution - which characterizes reality), BUT over half of the population always has expected incomes above average (negatively skewed expected income distribution - which, if true, will lead a strict majority to oppose redistribution). We seek restrictions on the parameters such that

(i) Next period’s income $y_{i+1}^t$ is stochastically increasing in current income $y_i^t$; (i.e. $\Pi$ is a monotone transition matrix)

(ii) The median income level is $a_2: \pi_1 < \pi_2 < \pi_1 + \pi_2$;

(iii) The median agent is poorer than the mean: $a_2 < \mu = \pi_1 a_1 + \pi_2 a_2 + (1 - \pi_1 - \pi_2) a_3$;

(iv) The median agent has expected income above the mean: $E[y_{i+1}^t | y_i^t = a_2] > \mu$.

NOTE: $M = [m_{kl}]_{3 \times 3}$ is a monotone transition matrix if row $k+1$ stochastically dominates row $k$: $m_{11} \geq m_{21} \geq m_{31}$ and $m_{11} + m_{12} \geq m_{21} + m_{22} \geq m_{31} + m_{32}$.

Also note that conditions (b) and (c) together ensure that a strict majority of the population would vote for current redistribution, while (b) and (d) together imply that a strict majority will vote against future redistribution.
a) What are the (weakest) restrictions on the parameters needed for the above four properties to be satisfied?

b) From US data, we estimate that $p = .55$, $q = .6$, $r = .5$, and $s = .7$; and $(a_1, a_2, a_3) = (16000, 36000, 91000)$. Does this satisfy the parametric restriction that you found in part a? Can we then conclude that the POUM holds in the US?
2. A problem in infinite-dimensions: Let $S = \{s_1, s_2, \ldots\}$ and let
\[
\Pi = \begin{bmatrix}
0 & 1 & 0 & 0 & \ldots \\
0 & 0 & 1 & 0 & \ldots \\
0 & 0 & 0 & 1 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}.
\]

What happens to the sequence $\Pi^k$ as $k$ becomes large? What are the ergodic sets? What is the invariant distribution? What is the issue here? (Relate this to the finite dimensional case)
3. Growth with Externality: Consider an economy with infinitely many agents each with preferences given by the utility function

$$\sum_{t=0}^{\infty} \beta^t \frac{c_t^\gamma}{\gamma} \quad 0 < \beta < 1 \quad \gamma < 1.$$ 

Each agent has one unit of time available in each period and can produce consumption goods according to

$$c_t = \alpha h_t^\theta h_t^{1-\theta} u_t,$$

where $0 < \theta < 1$, $\alpha > 0$, $u_t$ is the time spent in consumption goods production by the agent and $h_t$ denotes the average human capital across all agents in the economy. The externality is such that each agent’s productivity increases with the amount of human capital that other agents have. An agent’s human capital evolves according to

$$h_{t+1} = \delta h_t (1 - u_t),$$

where $\delta > 0$. Assume that each agent has the same initial stock of human capital.

a. Set up the social planner’s problem as a dynamic program. Note that the social planner internalizes the externality and treats $\overline{h}_t = h_t$.

b. Solve for the optimal growth rates of human capital and consumption, and solve for $u_t$.

c. Set up the representative agent’s dynamic programming problem and solve for the competitive equilibrium. Also, solve for a balanced growth path of the economy.
4. Monopolist’s problem: Consider this monopolistic version of the industry equilibrium model. A monopolist chooses output over time in order to maximize the present discounted value of its profits. Profits are defined as revenue minus cost, where the only cost that the monopolist incurs is the one associated with changing its level of production. Formally, the problem of the monopolist in sequence form is:

$$\max_{\{y_t\}_{t=1}^{\infty}} \sum_{t=0}^{\infty} \beta^t \Pi_t,$$

with $$y_0$$ given where $$0 < \beta < 1$$ denotes the monopolist’s discount factor. Profits, denoted by $$\Pi_t$$ and are given by

$$\Pi_t = p_t y_t - \frac{1}{2} (y_{t+1} - y_t)^2.$$

The price $$p_t$$ is given by the inverse demand function

$$p_t = a_0 - a_1 y_t,$$

where $$a_0 > 0$$ and $$a_1 > 1$$. Notice that the market price will exclusively depend on the production decision of the monopolist. The timing of the problem is as follows: at the beginning of period $$t$$ the monopolist decides on the quantity $$y_{t+1}$$ it wants to sell next period (at the price $$p_{t+1}$$). Revenue at time $$t$$ is collected at the beginning of the period and is determined by the production choices made in period $$t - 1$$.

a. Write down this problem as a dynamic programming problem.

b. Prove that there exists a unique value function $$V$$ that satisfies the functional equation and that this value function corresponds to the supremum of present discounted utilities in the sequence problem. Be careful to specify the complete metric space within which you look for solutions to the functional equation.

c. Write down the first order condition for the monopolists’ problem. Use it to show that the policy function $$y_t = g(y)$$ is strictly increasing in $$y$$. (Remember that to be able to apply the Benveniste-Scheinkman theorem and take the derivative of the value function you need to show that it is concave).
5. Linear Utility: Consider the case of a planner or an economically self-sufficient household (Robinson Crusoe?). The agent faces the following problem:

$$\max \sum_{t=0}^{\infty} \left( \frac{1}{2} \right)^t c_t$$

where $c_t$ represents consumption and $\frac{1}{2}$ is the discount factor. Output may be used either for consumption or capital formation. Output is produced with capital, according to $y_t = 4k_t - k_t^2$. Output installed as capital in period $t$ incurs an adjustment cost equal to $\frac{1}{2} \left( k_{t+1} - k_t \right)^2$.

a. What is the Bellman equation associated with the above decision problem? Prove the differentiability of the value function. Outline a clear argument.

b. Derive a closed form solution for the value and policy functions.

c. Calculate the steady state value of capital.
6. Growth: Consider an economy where a single homogeneous good is traded. The good can be used for consumption or for investment. The economy is populated by a large number of identical agents. Life-time utility of a representative agent is given by

$$E_0 \sum_{t=0}^{\infty} \beta^t \ln C_t + \gamma (1 - N_t), \ 0 < \beta < 1$$

where $N_t$ is the time spent working. Output is produced according to $Y_t = A_t K_t^\alpha N_t^{1-\alpha}, 0 < \alpha < 1$. $A_t$ is a stochastic productivity shock. Assume that

$$\ln A_t = \rho \ln A_{t-1} + \varepsilon_t, \varepsilon_t \sim iid(0, \sigma^2)$$

Further assume that $\varepsilon_t$ is observed before any period-$t$ decisions are undertaken. Physical capital depreciates at rate $\delta$.

(i) What is the Bellman’s equation that corresponds to the above problem. What are the states and controls?

(ii) Derive the Euler equation and the FOC for the fraction of time spent.

(iii) One of the ‘stylized facts’ over the last hundred years in the United States is that while GDP per capita has been growing at a constant rate, fraction of life-time spent working has been decreasing. Now assume that $\delta = 1$. Is there any hope for this model to match the above ‘fact’? In answering this question you could alter the manner in which productivity evolves. Explain and account for it clearly.

(iv) Now, assume that $\gamma = 0$ and $A_t = A$ for all $t$. If $\delta < 1$, can the value function take the form $V(K) = E + F \log(K)$ for constants $E$ and $F$? Why?
7. Markov Chains: Following Markov matrix is defines over the set \( S = \{\sigma, -\sigma\} \),

\[
\Pi = \begin{bmatrix}
\pi & 1 - \pi \\
1 - \pi & \pi
\end{bmatrix}
\]

Find the long run expected value \( E(s) \) of \( s \), variance \( Var(s) \) of \( s \), autocovariance \( E(ss') \) of \( s \), and autocorrelation \( \rho \) of \( s \).
8. Differentiability: Consider the following dynamic programming problem

\[ V(a, z_i) = \max_{c, a'} \left\{ U(c) + \beta \sum_{j=1}^{n} \pi_{ij} V(a', z_j) \right\}, \]

subject to

\[ c + a' = z_i + (1 + r)a, \]

and

\[ a' \geq 0. \]

Let \( U \) be a bounded, strictly increasing, strictly concave, continuously differentiable function. Suppose that \( 0 < r < \beta < 1 \). The bounded positive random variable \( z \) follows a \( n \)-point Markov process. In particular, \( z \) is drawn from the discrete set \( \mathcal{Z} \equiv \{z_1, z_2, \ldots, z_n\} \) according to the probability distribution specified by \( \pi_{ij} = \text{prob} \{ z' = z_j \mid z = z_i \} \), where \( 0 \leq \pi_{ij} \leq 1 \), and \( \sum_{j=1}^{n} \pi_{ij} = 1 \).

a) Is the value function \( V(a, z) \) is continuously differentiable in \( a \), for all \( a > 0 \), whenever \( a' > 0 \)?

b) Is the value function \( V(a, z) \) is continuously differentiable in \( a \), for all \( a > 0 \), whenever \( a' = 0 \)? What is the issue here?
9. Ergodic Sets: Suppose

\[ \Pi = \begin{bmatrix} \Pi_1 & 0 \\ 0 & \Pi_2 \end{bmatrix}, \]

where \( \Pi_1 \) and \( \Pi_2 \) are \( k \times k \) and \((l - k) \times (l - k)\) Markov matrices. Let

\[ \Pi_1 = \Pi_2 = \begin{bmatrix} 3/4 & 1/4 \\ 1/4 & 3/4 \end{bmatrix}. \]

(i) What is (are) the ergodic set(s)?
(ii) What is (are) the invariant distribution(s)?