Econ 714 Problem Set #4

Due by 1:00p CDT on Friday, April 18, 2008.

You may work together in a group of 2, 3 or 4. However, please hand in your own write-up, and name your collaborators.

I Optimal Consumption and Saving

Fix a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ and an information filtration $\{\mathcal{F}_t\}_{t=0}^{\infty}$, and suppose that a consumer's income at t is:

$$y_t = \phi + \sigma\epsilon_t, \quad \forall t \geq 1,$$

where $\sigma > 0$, y_0 is given, and ϵ_t's are i.i.d. standard normal innovations. The consumer maximizes:

$$U(c) = \mathbb{E}\left[\sum_{t=0}^{\infty}\left(\frac{1}{1+\delta}\right)^t u(c_t)\right],$$

where $\delta > 0$ is his subjective discount rate, and $u(\cdot)$ is CARA utility. That is,

$$u(c_t) = -\frac{1}{\theta}\exp\{-\theta c_t\}.$$

The consumer can borrow or save at a constant interest rate $r > 0$, and his sequential budget constraints are:

$$A_{t+1} = (1+r)A_t + y_t - c_t, \quad \forall t \geq 0,$$

where A_t denotes his asset holdings. A_0 is given. Assume that the natural borrowing constraint is in place, but disregard corner solutions in your analysis.

1. Write down the Bellman equation for this consumer's problem. In particular, let the value function (V) depend on A and y. Denote the next period variables with a prime; e.g. A_{t+1} becomes A'.

2. It is known that the value function looks like:

$$V(A, y) = -\frac{1}{\theta r}\exp\{-\theta r(A + py + q)\}, \tag{1}$$

   where p and q are coefficients to be determined. Show that the optimal consumption rule is linear in the state variables. To answer this question, you do not have to uncover what p and q actually are. Moreover, invoking the Benveniste-Scheinkman theorem may facilitate your derivation.

3. Express the optimal consumption rule as a function of state variables and fundamental parameters only. That is, substitute out p and q. If you are on the right track, you will run into $\mathbb{E}\exp\{-\theta rp\sigma\epsilon'\}$. Do not try to compute this. Simply call it $m(r, \sigma)$ and move on.

4. From your optimal consumption rule, now derive the optimal saving rule, $A' - A$, using the budget constraint. Show that the saving rule has three components: one with $(y - \phi)$, another with $\log\left(\frac{1+\delta}{1+r}\right)$, and the other with $\log m(r, \sigma)$. It is known that $\frac{\partial m}{\partial \sigma} > 0$. These components are called, not necessarily in the same order, dis-saving due to impatience, permanent-income hypothesis component, and precautionary saving. Match the names and the components. Briefly explain why the given names are appropriate.

II General Equilibrium

From the last question, let $\Psi(r) = \log\left(\frac{1+\delta}{1+r}\right)$ and $\Pi(r) = \log m(r, \sigma)$. Assume (correctly) that $m(r, \cdot) > 1$ for $r > 0$. Also, $m(0, \cdot) = 1$, which is easily verifiable.

Assume that the economy is populated by a continuum of ex ante identical but ex post heterogeneous agents of measure one. Each agent solves the problem in Section I. The risk-free asset is the pure consumption loan, and is in zero net supply. The initial cross-section distribution of income and asset is assumed to be its stationary distribution, $\Phi(A, y)$. By the law of large numbers, provided that we construct the space of agents and the probability space appropriately and assume pairwise independence of incomes, we will have an invariant cross-section distribution of income and asset, $\Phi(A, y)$. Under this invariant distribution, all the aggregate quantities remain constant.

1. Under the invariant distribution, the aggregate saving ($\int (A' - A) d\Phi$) in the economy must be zero. Why?

2. With regard to the invariant distribution Φ, what is the cross-sectional mean of $y - \phi$? Can we ignore this component in the determination of aggregate saving under the invariant distribution?

3. Express the aggregate saving function under the invariant distribution in terms of Π and Ψ. Prove that there exists no equilibrium with $r > \delta$.

4. Prove that there is at least one equilibrium with the interest rate r^* such that $0 < r^* < \delta$.