1.
Detailed solution omitted. KS statistic is expected to decrease as the sample size increases. Need to clearly demonstrate the value of the KS statistic for different sample sizes and for different distributions.

2.
We prove the following lemma:

**Lemma 1.** Suppose \( \{Y_{n,1}\}_{n=1}^{\infty} \) and \( \{Y_{n,2}\}_{n=1}^{\infty} \) are two sequences of random vectors such that \( Y_{n,1} \in \mathbb{R}^k \) and \( Y_{n,2} \in \mathbb{R}^l \) and as \( n \to \infty \), \( Y_{n,1} \to_d Y_1 \) and \( Y_{n,2} \to_p r \) for a random vector \( Y_1 \in \mathbb{R}^k \) and a constant vector \( r \in \mathbb{R}^l \). Then,

\[
\begin{pmatrix}
Y_{n,1} \\
Y_{n,2}
\end{pmatrix} \to_d
\begin{pmatrix}
Y_1 \\
r
\end{pmatrix}.
\]

Take a constant vector \( c = (c'_1, c'_2) \in \mathbb{R}^k \times \mathbb{R}^l \). Since a linear combination is a continuous function, we have \( c'_1 Y_{n,1} \to_d c'_1 Y_1 \) and \( c'_2 Y_{n,2} \to_d c'_2 r \) by CMT. Note that \( c'_1 Y_{n,1} \in \mathbb{R}^k \) and \( c'_2 Y_{n,2} \in \mathbb{R}^l \), i.e., they are random variables. This implies that we can use the scalar version of the Lemma 1:

\[
\begin{pmatrix}
c'_1 Y_{n,1} \\
c'_2 Y_{n,2}
\end{pmatrix} \to_d
\begin{pmatrix}
c'_1 Y_1 \\
c'_2 r
\end{pmatrix}.
\]

By CMT, \( c'_1 Y_{n,1} + c'_2 Y_{n,2} \to_d c'_1 Y_1 + c'_2 r \). We can rewrite it as

\[
c'( Y_{n,1} Y_{n,2}) = c'_1 Y_{n,1} + c'_2 Y_{n,2} \to_d c'_1 Y_1 + c'_2 r = c'
\begin{pmatrix}
Y_1 \\
r
\end{pmatrix}.
\]

Since the vector \( c \) is arbitrary, by Cramer-Wold device, we conclude

\[
\begin{pmatrix}
Y_{n,1} \\
Y_{n,2}
\end{pmatrix} \to_d
\begin{pmatrix}
Y_1 \\
r
\end{pmatrix}.
\]

3.
Without loss of generality, assume \( EX_{1i} = EX_{2i} = 0 \) throughout the question.

(a) I propose the following estimator for \( \rho \):

\[
\hat{\rho} = \frac{n^{-1} \sum_{i=1}^{n} (X_{1i} - \bar{X}_{1n})(X_{2i} - \bar{X}_{2n})}{\left(n^{-1} \sum_{i=1}^{n} (X_{1i} - \bar{X}_{1n})^2\right)^{1/2} \left(n^{-1} \sum_{i=1}^{n} (X_{2i} - \bar{X}_{2n})^2\right)^{1/2}}
= \frac{n^{-1} \sum_{i=1}^{n} X_{1i}X_{2i} - \bar{X}_{1n}\bar{X}_{2n}}{\left(n^{-1} \sum_{i=1}^{n} X_{1i}^2 - (\bar{X}_{1n})^2\right)^{1/2} \left(n^{-1} \sum_{i=1}^{n} X_{2i}^2 - (\bar{X}_{2n})^2\right)^{1/2}}.
\]
Thus, \( \hat{\rho} \) by multivariate CLT,

\[
\rho = \frac{\text{Cov}(X_{1i}, X_{2i})}{\text{Var}(X_{1i})} = \rho_1 \sigma_2,
\]

for \( j = 1, 2 \).

Consider a function \( g : \mathbb{R}^5 \to \mathbb{R} \) such that \( g(a, b, c, d, e) = \frac{e - ab}{(c-a^2)(d-b^2)^{1/2}} \). Then

\[
\hat{\rho} = g \left( \bar{X}_{1n}, \bar{X}_{2n}, n^{-1} \sum_{i=1}^{n} X_{1i}^2, n^{-1} \sum_{i=1}^{n} X_{2i}^2, n^{-1} \sum_{i=1}^{n} X_{1i}X_{2i} \right).
\]

Since \( g(a, b, c, d, e) \) is continuous where \( c - a^2 > 0 \) and \( d - b^2 > 0 \), by CMT, we have

\[
\hat{\rho} = g \left( \bar{X}_{1n}, \bar{X}_{2n}, n^{-1} \sum_{i=1}^{n} X_{1i}^2, n^{-1} \sum_{i=1}^{n} X_{2i}^2, n^{-1} \sum_{i=1}^{n} X_{1i}X_{2i} \right) \to_p g(EX_{1i}, EX_{2i}, EX_{1i}^2, EX_{2i}^2, EX_{1i}X_{2i}) = \rho.
\]

Thus, \( \hat{\rho} \) is consistent for \( \rho \).

(b) By multivariate CLT,

\[
\sqrt{n} \left( \begin{pmatrix} \bar{X}_{1n} \\ \bar{X}_{2n} \\ n^{-1} \sum_{i=1}^{n} X_{1i}^2 \\ n^{-1} \sum_{i=1}^{n} X_{2i}^2 \\ n^{-1} \sum_{i=1}^{n} X_{1i}X_{2i} \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ \sigma_1^2 \\ \sigma_2^2 \\ \rho \sigma_1 \sigma_2 \end{pmatrix} \right) \to_d N(0, \Sigma),
\]

where \( \mathbf{0} \) is \( 5 \times 1 \) vector of zeros and \( \Sigma \) is \( 5 \times 5 \) covariance matrix. Let

\[
G(a, b, c, d, e) = \begin{pmatrix} \frac{\partial g(a, b, d, c, e)}{\partial a} & \frac{\partial g(a, b, d, c, e)}{\partial b} & \frac{\partial g(a, b, d, c, e)}{\partial c} & \frac{\partial g(a, b, d, c, e)}{\partial d} & \frac{\partial g(a, b, d, c, e)}{\partial e} \end{pmatrix}.
\]

You should check whether \( G(0, 0, \sigma_1, \sigma_2, \rho \sigma_1 \sigma_2) \) exists and is non-zero to apply the delta method. Write \( G(0, 0, \sigma_1, \sigma_2, \rho \sigma_1 \sigma_2) = \mathbb{G} \). By the delta method,

\[
\sqrt{n}(\hat{\rho} - \rho) = \sqrt{n} \left( \begin{pmatrix} \bar{X}_{1n} \\ \bar{X}_{2n} \\ n^{-1} \sum_{i=1}^{n} X_{1i}^2 \\ n^{-1} \sum_{i=1}^{n} X_{2i}^2 \\ n^{-1} \sum_{i=1}^{n} X_{1i}X_{2i} \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ \sigma_1^2 \\ \sigma_2^2 \\ \rho \sigma_1 \sigma_2 \end{pmatrix} \right) \to_d N(0, G \Sigma G') .
\]

Note that \( G \Sigma G' \) is a scalar and is the asymptotic variance of \( \sqrt{n}(\hat{\rho} - \rho) \).

\footnote{This is not a necessary condition for the WLLN, though.}
4. If \( Y_n \sim b(n, p) \) then \( Y_n = \sum_{i=1}^{n} X_i \) with \( X_i \sim Bern(p) \) for \( i = 1, ..., n \). Since \( E X_i = p \), by the WLLN,
\[
\frac{Y_n}{n} = \frac{1}{n} \sum_{i=1}^{n} X_i \rightarrow_p E X_i = p.
\]
Let \( g(x) = x(1 - x) \). Then \( g(x) \) is continuous. By CMT (or Slutsky’s theorem),
\[
\frac{Y_n}{n} \left(1 - \frac{Y_n}{n}\right) = g \left(\frac{Y_n}{n}\right) \rightarrow_p g(p) = p(1 - p).
\]

5. Since \( X_i \)'s are random sample from \( \chi^2(50) \), \( EX_i = 50 \) and \( Var(X_i) = 100 \). CLT implies that \( \sqrt{n}(\bar{X} - EX_i)/10 \) is asymptotically standard normal. We use this fact to approximate the probability \( P(49 < \bar{X} < 51) \):
\[
P(49 < \bar{X} < 51) = P(-1 < \bar{X} - 50 < 1)
= P(\sqrt{100} \times -1 < \sqrt{100}(\bar{X} - 50) < \sqrt{100} \times 1)
= P(-1 < \sqrt{100}(\bar{X} - 50)/10 < 1)
\approx P(-1 < Z < 1) = 2\Phi(1) - 1 = 0.6826.
\]

6. Let \( X_i \)'s be random sample. Then
\[
P(X_i < 3) = \int_{1}^{3} \frac{1}{x^2} dx = \frac{2}{3}.
\]
Each observation is either larger than (or equal to) 3 with probability 1/3 or less than 3 with probability 2/3. Let \( Y_i \) be a random variable such that \( Y_i = 1\{X_i < 3\} \). Then \( Y_i \) is a Bernoulli random variable with \( p = 2/3 \) and \( \sum_{i=1}^{72} Y_i \) is a binomial random variable such that \( \sum_{i=1}^{72} Y_i \sim b(72, 2/3) \). Let \( Y_n = \sum_{i=1}^{72} Y_i \). The distribution of \( Y_n \) can be approximated by \( Z_n \) such that \( Z_n \sim N(48, 16) \): \( P(Y_n > 50) \approx P(Z_n > 50.5) \) with continuity correction. Now
\[
P(Y_n > 50) \approx P(Z_n > 50.5)
= P\left(\frac{Z_n - 48}{4} > \frac{50.5 - 48}{4}\right)
= P(Z > 0.625) = 0.266,
\]
where \( Z \) is standard normal random variable.
7.

First, we show $Y_1 \to_P a$. Take any $\varepsilon > 0$.

\[
P(|Y_1 - a| < \varepsilon) = P(a - \varepsilon < Y_1 < a + \varepsilon) = P(a < Y_1 < a + \varepsilon) = P(a < \min X_i < a + \varepsilon) = 1 - P(\min X_i \geq a + \varepsilon) = 1 - P(X_i \geq a + \varepsilon, i = 1, 2, \ldots, n) = 1 - (P(X_i \geq a + \varepsilon))^n = 1 - \left(\frac{b - a - \varepsilon}{b - a}\right)^n \to 1,
\]

as $n \to \infty$. Similarly, we can show $Y_2 \to_P b$. In class, we learned that if $Y_{1n} \to_d Y_1$ and $Y_{2n} \to_d r$, where $r$ is a constant, then $(Y_{1n}, Y_{2n})' \to_d (Y_1, r)'$. This is Lemma 1. Convergence in probability to a constant is the same as convergence in distribution to a constant. This implies that we have $(Y_1, Y_2)' \to_P (a, b)'$ by Lemma 1.

8.

By Lemma 1, we have

\[
\left(\begin{array}{c}
X_n - Y_n \\
X_n
\end{array}\right) \to_d \left(\begin{array}{c}
0 \\
X
\end{array}\right).
\]

By CMT, $X_n - (X_n - Y_n) = Y_n \to_d X - 0 = X$. 

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