Large Deviations and Stochastic Stability in Single and Double Limits

William H. Sandholm
University of Wisconsin

based on joint work with Mathias Staudigl (Bielefeld)
Michel Benaïm (Neuchâtel)
Population Games, Revision Protocols, and Stochastic Evolutionary Dynamics

Population games model strategic interactions among large numbers of small, anonymous agents.

To model the dynamics of behavior, we use revision protocols to describe how myopic individual agents decide whether and when to switch strategies.

A population game $F$, a revision protocol $\sigma^\eta$ with noise level $\eta > 0$, and a population size $N$ define a Markov chain $X^{N,\eta}$ on the set of population states.
The process $X^{N, \eta}$ can be analyzed on a variety of time scales:

**Short/medium:** convergence to attractors.
Movement to stable equilibria, stable cycles, other attracting sets

**Long:** large deviations.
Transitions between stable equilibria/attracting sets

**Very long:** stochastic stability.
Equilibrium selection via stationarity properties
The process $X^{N,\eta}$ can be analyzed on a variety of time scales:

Short/medium: convergence to attractors.
Movement to stable equilibria, stable cycles, other attracting sets

Long: large deviations.
Transitions between stable equilibria/attracting sets

Very long: stochastic stability.
Equilibrium selection via stationarity properties
The process $X^{N,\eta}$ can be analyzed on a variety of time scales:

**Short/medium:** convergence to attractors.
Movement to stable equilibria, stable cycles, other attracting sets.

**Long:** large deviations.
Transitions between stable equilibria/attracting sets.

**Very long:** stochastic stability.
Equilibrium selection via stationarity properties.
The process $X^{N, \eta}$ can be analyzed on a variety of time scales:

**Short/medium:** convergence to attractors.
Movement to stable equilibria, stable cycles, other attracting sets

**Long:** large deviations.
Transitions between stable equilibria/attracting sets

**Very long:** stochastic stability.
Equilibrium selection via stationarity properties

This talk focuses on large deviations properties.
These are built upon convergence properties, and underlie analyses of stochastic stability.
Our analyses provide theories of **equilibrium breakdown** in population games.

Why study this?

**Theoretical motivation:**

The assumptions underlying disequilibrium learning also generate equilibrium breakdown and re-emergence.

**Applied motivations:**

In applications and experiments, equilibrium does break down!

Equilibrium breakdown via large deviations has been the basis for macroeconomic models (Sargent (1999), Cho et al. (2002), Williams (2014)).

There the breakdown is a consequence of exogenous noise, while here it arises endogenously.
Single and double limits

Analyses of $X^{N,\eta}$ are commonly performed as the parameters $N$ and $\eta$ approach their limiting values.

The small noise limit, $\eta \to 0$, emphasizes rarity of mistakes.

The large population limit, $N \to \infty$, emphasizes averaging effects of large numbers.

We show that these single limits allow for clean characterizations of large deviations properties, but ones that only allow special examples to be solved.

We then argue that by taking double limits ($\eta$ then $N$; $N$ then $\eta$), one obtains characterizations that seem tractable quite generally.
The Model

Population games

\[ S = \{1, \ldots , n\} \] strategies
\[ X = \{x \in \mathbb{R}^n_+ : \sum_{i \in S} x_i = 1\} \] population states/mixed strategies
\[ F_i : X \to \mathbb{R} \] payoffs to strategy \( i \)
\[ F : X \to \mathbb{R}^n \] payoffs to all strategies

Example: Matching in symmetric normal form games \( A \in \mathbb{R}^{n \times n} \):

\[ F_i(x) = \sum_{j \in S} A_{ij}x_j = (Ax)_i \]
\[ F(x) = Ax \]

Example: Potential games: \( F(x) = \nabla f(x) \) for some \( f : X \to \mathbb{R} \).

(\( \cap = \text{common interest games: } F(x) = Ax, A = A' \Rightarrow f(x) = \frac{1}{2}x'Ax \))
Example: A common interest game and its potential function

\[ F(x) = \begin{pmatrix} F_1(x) \\ F_2(x) \\ F_3(x) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}; \quad f(x) = \frac{1}{2} \left( (x_1)^2 + 2(x_2)^2 + 3(x_3)^2 \right). \]
Noisy best response protocols

Agents make decisions by applying a noisy best response protocol $\sigma^n$ with noise level $\eta$.

$\sigma^n : \mathbb{R}^n \rightarrow \text{int}(X)$, Lipschitz continuous

$\sigma_j^n(\pi)$ is the conditional switch probability to strategy $j$

In each period, a randomly chosen agent receives a revision opportunity, switching to strategy $j$ with probability $\sigma_j^n(F(x))$. 
Unlikelihood functions

For $\sigma^n$ to be a noisy best response protocol, we should have

$$\lim_{\eta \to 0} \sigma^n_i(\pi) > 0 \iff i \in \arg\max\limits_{j \in S} \pi_j.$$

We further assume that $\sigma^n$ admits a continuous unlikelihood function $\Upsilon: \mathbb{R}^n \to \mathbb{R}_+^n$:

$$\Upsilon_j(\pi) = -\lim_{\eta \to 0} \eta \log \sigma^n_j(\pi) \quad \text{(i.e., } \sigma^n_j(\pi) \approx \exp(-\eta^{-1} \Upsilon_j(\pi)) \text{).}$$

$\Upsilon_j(\pi)$ is the rate of decay of the probability that $j$ is chosen as $\eta \to 0$. 
Example: The logit protocol (Blume (1993))

\[
\sigma_j^\eta(\pi) = \frac{\exp(\eta^{-1}\pi_j)}{\sum_{k \in S} \exp(\eta^{-1}\pi_k)}.
\]

(⇒) \( \Upsilon_j(\pi) = \max_{k \in S} \pi_k - \pi_j. \)

Example: The probit protocol (Myatt and Wallace (2003))

\[
\sigma_j^\eta(\pi) = \mathbb{P}\left( j = \arg\max_{k \in S} (\pi_k + Z_k) \right), \quad \{Z_k\}_{k \in S} \text{ i.i.d. normal}(0, \eta).
\]

(⇒) \( \Upsilon_j \) is piecewise quadratic (Dokumaci and Sandholm (2011)).
Non-example: The best response with mutations (BRM) protocol 
(Kandori, Mailath, and Rob (1993), Young (1993))

\[ \sigma_j^\eta(\pi) = \begin{cases} 
1 - \eta & \text{if } j = \arg\max_{k\in S} \pi_k \\
\frac{1}{n-1} \eta & \text{otherwise} 
\end{cases} \]

(ignoring ties).

\[ \Upsilon_j(\pi) = 1_{\{j \notin \arg\max_{k\in S} \pi_k\}} \]

This protocol and its unlikelihood function are discontinuous.
(But we can handle this protocol separately.)
The Markov chain

A population game $F$, a revision protocol $\sigma^{\eta}$, and a population size $N$ define a Markov chain $X^{N,\eta} = \{X^{N,\eta}_k\}_{k=0}^{\infty}$ with period length $\frac{1}{N}$ on the finite state space $X^N = X \cap \frac{1}{N} \mathbb{Z}^n = \{x \in X : Nx \in \mathbb{Z}^n\}$. 
The evolution of $X^{N,\eta}$ is described recursively by

$$X_{k+1}^{N,\eta} = X_k^{N,\eta} + \frac{1}{N}\bar{\zeta}_{k+1}^{\eta},$$

where the normalized increment $\bar{\zeta}_{k+1}^{\eta}$ has conditional law

$$P\left(\bar{\zeta}_{k+1}^{\eta} = e_j - e_i \mid X_k^{N,\eta} = x\right) = x_i \sigma_j^{\eta}(F(x)) \text{ if } i \neq j.$$
Analyses of different time scales: heuristics

Short/medium: convergence to equilibrium
Analyses of different time scales: heuristics

Long: excursions between attractors via large deviations
Analyses of different time scales: heuristics

Long: excursions between attractors via large deviations

\[ \rightsquigarrow \text{Very long: stochastic stability.} \]
Stationary distributions and stochastic stability

Since $\sigma_j^\eta > 0$, the Markov chain $X^{N,\eta}$ is irreducible and aperiodic.

It therefore admits a unique stationary distribution $\mu^{N,\eta}$.

This distribution is the limiting distribution of the Markov chain, and its limiting empirical distribution on almost every sample path.

Both for tractability and to obtain sharp predictions, one typically considers the behavior of $\mu^{N,\eta}$ after certain limits are taken.

Then states retaining mass are said to be stochastically stable.

The asymptotics of $\mu^{N,\eta}$, and thus the identity of the stochastically stable states, can be deduced from the large deviations properties of $X^{N,\eta}$ (Freidlin and Wentzell (1984)).
Formulations of large deviations and stochastic stability

Taking different limits leads to different notions of stochastic stability:

\[
\lim_{\eta \to 0} \mu_{N,\eta} \quad \text{the small noise limit (SNL)}
\]

\[
\lim_{N \to \infty} \lim_{\eta \to 0} \mu_{N,\eta} \quad \text{the small noise double limit (SNDL)}
\]

\[
\lim_{N \to \infty} \mu_{N,\eta} \quad \text{the large population limit (LPL)}
\]

\[
\lim_{\eta \to 0} \lim_{N \to \infty} \mu_{N,\eta} \quad \text{the large population double limit (LPDL)}
\]
Formulations of large deviations and stochastic stability

Taking different limits leads to different notions of stochastic stability:

$$\lim_{\eta \to 0} \mu^{N,\eta}_{\eta}$$ \quad the small noise limit (SNL)

$$\lim_{N \to \infty} \lim_{\eta \to 0} \mu^{N,\eta}_{N}$$ \quad the small noise double limit (SNDL)

$$\lim_{N \to \infty} \mu^{N,\eta}_{N}$$ \quad the large population limit (LPL)

$$\lim_{\eta \to 0} \lim_{N \to \infty} \mu^{N,\eta}_{N}$$ \quad the large population double limit (LPDL)

SNL is the original approach to stochastic stability.

Kandori, Mailath, and Rob (1993), Young (1993)

It emphasizes the rarity of mistakes as the force behind selection.
Formulations of large deviations and stochastic stability

Taking different limits leads to different notions of stochastic stability:

\[ \lim_{\eta \to 0} \mu^{N,\eta} \quad \text{the small noise limit (SNL)} \]

\[ \lim_{N \to \infty} \lim_{\eta \to 0} \mu^{N,\eta} \quad \text{the small noise double limit (SNDL)} \]

\[ \lim_{N \to \infty} \mu^{N,\eta} \quad \text{the large population limit (LPL)} \]

\[ \lim_{\eta \to 0} \lim_{N \to \infty} \mu^{N,\eta} \quad \text{the large population double limit (LPDL)} \]

SNDL is SNL at arbitrarily large population sizes.
Formulations of large deviations and stochastic stability

Taking different limits leads to different notions of stochastic stability:

\[ \lim_{\eta \to 0} \mu_{N,\eta} \quad \text{the small noise limit (SNL)} \]

\[ \lim_{N \to \infty} \lim_{\eta \to 0} \mu_{N,\eta} \quad \text{the small noise double limit (SNDL)} \]

\[ \lim_{N \to \infty} \mu_{N,\eta} \quad \text{the large population limit (LPL)} \]

\[ \lim_{\eta \to 0} \lim_{N \to \infty} \mu_{N,\eta} \quad \text{the large population double limit (LPDL)} \]

The large population limit underlies deterministic mean dynamics.

Benaïm and Weibull (2003)

Here, LPL emphasizes the averaging effects of large numbers as the force governing transitions and selection.
Formulations of large deviations and stochastic stability

Taking different limits leads to different notions of stochastic stability:

\[ \lim_{\eta \to 0} \mu^{N,\eta} \quad \text{the small noise limit (SNL)} \]
\[ \lim_{N \to \infty} \lim_{\eta \to 0} \mu^{N,\eta} \quad \text{the small noise double limit (SNDL)} \]
\[ \lim_{N \to \infty} \mu^{N,\eta} \quad \text{the large population limit (LPL)} \]
\[ \lim_{\eta \to 0} \lim_{N \to \infty} \mu^{N,\eta} \quad \text{the large population double limit (LPDL)} \]

LPDL is LPL at arbitrarily small noise levels.
Formulations of large deviations and stochastic stability

Taking different limits leads to different notions of stochastic stability:

\[
\lim_{\eta \to 0} \mu^{N,\eta}_{\eta} \quad \text{the small noise limit (SNL)}
\]

\[
\lim_{N \to \infty} \lim_{\eta \to 0} \mu^{N,\eta}_{\eta} \quad \text{the small noise double limit (SNDL)}
\]

\[
\lim_{N \to \infty} \mu^{N,\eta}_{\eta} \quad \text{the large population limit (LPL)}
\]

\[
\lim_{\eta \to 0} \lim_{N \to \infty} \mu^{N,\eta}_{\eta} \quad \text{the large population double limit (LPDL)}
\]

All formulations have been analyzed exhaustively when \( n = 2 \).
Binmore, Samuelson, and Vaughan (1995), Binmore and Samuelson (1997),

When \( n = 2 \), under the noisy best response models studied here,
the asymptotics of \( \mu^{N,\eta} \) in the two double limits are identical.
Sandholm (2010)
Formulations of large deviations and stochastic stability

Taking different limits leads to different notions of stochastic stability:

\[
\lim_{\eta \to 0} \mu_{N,\eta} \quad \text{the small noise limit (SNL)}
\]

\[
\lim_{N \to \infty} \lim_{\eta \to 0} \mu_{N,\eta} \quad \text{the small noise double limit (SNDL)}
\]

\[
\lim_{N \to \infty} \mu_{N,\eta} \quad \text{the large population limit (LPL)}
\]

\[
\lim_{\eta \to 0} \lim_{N \to \infty} \mu_{N,\eta} \quad \text{the large population double limit (LPDL)}
\]

All formulations have been analyzed exhaustively when \( n = 2 \).
Binmore, Samuelson, and Vaughan (1995), Binmore and Samuelson (1997),

When \( n = 2 \), under the noisy best response models studied here,
the asymptotics of \( \mu_{N,\eta} \) in the two double limits are identical.
Sandholm (2010)
Formulations of large deviations and stochastic stability

Taking different limits leads to different notions of stochastic stability:

\[ \lim_{\eta \to 0} \mu_{N,\eta} \] the small noise limit (SNL)

\[ \lim_{N \to \infty} \lim_{\eta \to 0} \mu_{N,\eta} \] the small noise double limit (SNDL)

\[ \lim_{N \to \infty} \mu_{N,\eta} \] the large population limit (LPL)

\[ \lim_{\eta \to 0} \lim_{N \to \infty} \mu_{N,\eta} \] the large population double limit (LPDL)

Under the BRM protocol, SNL and SNDL are well understood.
Formulations of large deviations and stochastic stability

Taking different limits leads to different notions of stochastic stability:

\[
\lim_{\eta \to 0} \mu_{N,\eta} \quad \text{the small noise limit (SNL)}
\]

\[
\lim_{N \to \infty} \lim_{\eta \to 0} \mu_{N,\eta} \quad \text{the small noise double limit (SNDL)}
\]

\[
\lim_{N \to \infty} \mu_{N,\eta} \quad \text{the large population limit (LPL)}
\]

\[
\lim_{\eta \to 0} \lim_{N \to \infty} \mu_{N,\eta} \quad \text{the large population double limit (LPDL)}
\]

For other protocols, SNL is understood in principle, but is intractable unless \(N\) is small.
Formulations of large deviations and stochastic stability

Taking different limits leads to different notions of stochastic stability:

\[ \lim_{\eta \to 0} \mu_{N,\eta} \] the small noise limit (SNL)

\[ \lim_{N \to \infty} \lim_{\eta \to 0} \mu_{N,\eta} \] the small noise double limit (SNDL)

\[ \lim_{N \to \infty} \mu_{N,\eta} \] the large population limit (LPL)

\[ \lim_{\eta \to 0} \lim_{N \to \infty} \mu_{N,\eta} \] the large population double limit (LPDL)

When \( n > 2 \), little is known about

(i) SNDL, except under BRM

(ii) LPL or LPDL under any protocols
Outline

1. Review: the small noise limit.

2. The small noise double limit.
   2.1 We characterize long run behavior in terms of solutions to nonsmooth optimal control problems.
   2.2 We solve these problems explicitly in a basic class of examples (certain three-strategy coordination games under the logit protocol).

3. The large population limit.
The Small Noise Limit

How does the process $X^{N,\eta}$ behave as $\eta$ becomes small?

This analysis follows standard lines (Freidlin and Wentzell (1984), Young (1998), Catoni (1999)).
The discrete best response dynamic and its recurrent classes

What are the “typical” behaviors of $X^{N, \eta}$ when $\eta$ is small?

Suppose that in each discrete time period, a single agent is chosen at random, and that he switches to a best response. Then the state follows a solution to the discrete best response dynamic

\[
(D) \quad x_{k+1}^{N} - x_{k}^{N} \in \frac{1}{N} V^{N}(x_{k}^{N}),
\]

where $V^{N}(x) = \left\{ e_{j} - e_{i} : x_{i} > 0 \text{ and } j \in \arg\max_{k \in S} F_{k}(x) \right\}$.

$K^{N} \subseteq X^{N}$ is strongly invariant if no solution to (D) starting in $K^{N}$ exits $K^{N}$. $K^{N}$ is a recurrent class if it is a minimal strongly invariant set.

When $\eta$ is small, $X^{N, \eta}$ spends most periods in recurrent classes. Below we assume for convenience that $K^{N} = \{ x^{*} \} \in \{ e_{1}, \ldots, e_{n} \}$. 
The discrete best response dynamic and its recurrent classes
Recurrent classes and transition paths
Step costs, path costs, and transition costs in the small noise limit

The cost of a step from \( x \in \mathcal{X}^N \) to \( y = x + \frac{1}{N}(e_j - e_i) \in \mathcal{X}^N \) is

\[
c_{x,y}^N = -\lim_{\eta \to 0} \eta \log \mathbb{P}(X_{k+1}^{N,\eta} = y \mid X_k^{N,\eta} = x) \\
= -\lim_{\eta \to 0} \eta \log (x_i \sigma_j^\eta(F(x))) \\
= \Upsilon_j(F(x)) \quad \text{(i.e., } \mathbb{P}(X_{k+1}^{N,\eta} = y \mid X_k^{N,\eta} = x) \approx \exp(-\eta^{-1}c_{\phi_k^N,\phi_{k+1}^N}^N) \text{)}.
\]

The cost of path \( \phi^N = \{\phi_k^N\}_{k=0}^{\ell^N} \) of finite length \( \ell^N \) is

\[
c^N(\phi^N) = \sum_{k=0}^{\ell^N-1} c_{\phi_k^N,\phi_{k+1}^N}^N.
\]

It is the rate of decay of the probability of the path as \( \eta \) vanishes:

\[
\mathbb{P}(X_k^{N,\eta} = \phi_k^N, k = 0, \ldots, \ell^N \mid X_0^{N,\eta} = \phi_0^N) \approx \prod_{k=0}^{\ell^N-1} \exp(-\eta^{-1}c_{\phi_k^N,\phi_{k+1}^N}^N) \\
= \exp(-\eta^{-1}c^N(\phi^N)).
\]
Transition costs and limiting stationary distributions

Let $\Phi^N(x^*,y^*)$ be the set of paths through $X^N$ from $x^*$ to $y^*$.

The cost of a transition from $x^*$ to $y^*$ is

$$C^N(x^*,y^*) = \min \left\{ c^N(\phi^N) : \phi^N \in \Phi^N(x^*,y^*) \right\}.$$  

It is the rate of decay of the probability of a transition from $x^*$ to $y^*$.

The limiting stationary distribution $\lim_{\eta \to 0} \mu^{N,\eta}$ can be computed from the transition costs $C^N(x^*,y^*)$.

The key step is to construct minimum cost trees on the set of recurrent classes.
Why the small noise limit is not tractable

\[ c_{x,y}^N = \Upsilon_j(F(x)), \quad \text{where } y = x + \frac{1}{N}(e_j - e_i) \]

\[ C^N(x^*,y^*) = \min \left\{ \sum_{k=0}^{\ell^N-1} c_{\phi^N_k,\phi^N_{k+1}}^N : \phi^N \in \Phi^N(x^*,y^*) \right\}. \]

The transition cost \( C^N(x^*,y^*) \) is the solution to a discrete optimal control problem.

It is not tractable except

(i) under BRM,
(ii) when \( n = 2 \), or
(iii) when \( N \) is quite small.
The Small Noise Double Limit

By taking a second limit in $N$, we can try to replace discrete control problems with continuous ones.
The Small Noise Double Limit

Let $\phi^N = \{\phi^N_k\}_{k=0}^{\ell^N}$ be a path for the $N$-agent process.

We can express its cost as a discrete path integral:

$$c^N(\phi^N) = \sum_{k=0}^{\ell^N-1} c^N_{\phi^N_k, \phi^N_{k+1}}$$

$$= \sum_{k=0}^{\ell^N-1} \Upsilon^N_k(F(\phi^N_k)) \quad \text{(where } \phi^N_{k+1} - \phi^N_k = \frac{1}{N}(e^N_{j,k} - e^N_{i,k}))$$

$$= \sum_{k=0}^{\ell^N-1} \langle \Upsilon(F(\phi^N_k)), [\dot{\phi}^N_k]_+ \rangle,$$

where $\dot{\phi}^N_k = N(\phi^N_{k+1} - \phi^N_k)$ is the discrete right derivative.

This suggests that we define the cost of continuous path $\phi$ by

$$c(\phi) = \int_0^T \langle \Upsilon(F(\phi_t)), [\dot{\phi}_t]_+ \rangle \, dt.$$
\[ c^N(\phi^N) = \sum_{k=0}^{\ell^N-1} \langle \Upsilon(F(\phi^N_k)), [\dot{\phi}^N_k]_+ \rangle \]

\[ c(\phi) = \int_0^T \langle \Upsilon(F(\phi_t)), [\dot{\phi}_t]_+ \rangle \, dt \]

a discrete path through \( X^N \)  
a continuous path through \( X \)

To justify the definition of \( c(\phi) \), we must prove that \( c(\phi) \) approximates \( c^N(\phi^N) \) in an appropriate sense.
Convergence of transition costs

The cost of a transition from $x^*$ to $y^*$ for fixed $N$ is

$$C^N(x^*, y^*) = \min \left\{ c^N(\phi^N) : \phi^N \in \Phi^N(x^*, y^*) \right\}.$$  

Define $C(x^*, y^*) = \min \left\{ c(\phi) : \phi \in \Phi(x^*, y^*) \right\}.$

**Theorem: Convergence of transition costs**

Under mild regularity conditions,  
$$\lim_{N \to \infty} \frac{1}{N} C^N(x^*, y^*) = C(x^*, y^*).$$
Convergence of transition costs

The cost of a transition from $x^*$ to $y^*$ for fixed $N$ is

$$C_N(x^*, y^*) = \min \left\{ c_N(\phi^N) : \phi^N \in \Phi^N(x^*, y^*) \right\}.$$ 

Define $C(x^*, y^*) = \min \left\{ c(\phi) : \phi \in \Phi(x^*, y^*) \right\}$.

**Theorem:** Convergence of transition costs

*Under mild regularity conditions,* 

$$\lim_{N \to \infty} \frac{1}{N} C_N(x^*, y^*) = C(x^*, y^*).$$

Idea of the proof:

Lower bound ($\geq$): Given a discrete path $\phi^N$, define a continuous path $\phi^{(N)}$ via interpolation.

Upper bound ($\leq$): Given a continuous path $\phi$, explicitly construct an “approximating” discrete path $\phi^N$. 
Computing transition costs

Computing $C(x^*, y^*)$ means solving a multivariate control problem.

$$C(x^*, y^*) = \inf \left\{ \int_0^T L(\phi_t, \dot{\phi}_t) \, dt : \phi \in \Phi(x^*, y^*) \right\}.$$ 

However, $L(x, z) \equiv \langle \Upsilon(F(x)), [z]_+ \rangle$ is

(i) linearly homogeneous in $z$, and
(ii) linear in $z$ on each orthant of $\mathbb{R}^n$.

In some interesting settings, costs can be computed analytically.
We now consider evolution:

• under the logit rule: $\gamma_j(\pi) = \max_{k \in S} \pi_k - \pi_j$

• in linear games: $F(x) = Ax$ (not necessarily potential games!)

We focus on a setting whose general solution is easy to describe:

• three-strategy coordination games

• with the marginal bandwagon property (Kandori and Rob (1998))

• and an interior Nash equilibrium $x^*$
Solving the cost minimization problem via dynamic programming

We consider the problem of reaching $e_k$ from all other states.

Let $V : X \rightarrow \mathbb{R}_+$ be the minimized cost of reaching $e_k$.

The Hamilton-Jacobi-Bellman equation associated with $c(\cdot)$ is

$$0 = \inf_{u \in TX} \left( [u]' \gamma(F(x)) + DV(x)u \right).$$

A verification theorem (Boltyanskii (1966), Piccoli & Sussmann (2000)) shows that it is enough to find a solution of the HJB equation that is (i) continuous and (ii) continuously differentiable except on some lower-dimensional manifolds.

We explicitly construct such solutions.
Potential games: optimal transition paths and their costs

The costs of these optimal paths are given by decreases in potential.
Example: A common interest game and its potential function

\[
F(x) = \begin{pmatrix} F_1(x) \\ F_2(x) \\ F_3(x) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}; \quad f(x) = \frac{1}{2} \left( (x_1)^2 + 2(x_2)^2 + 3(x_3)^2 \right).
\]
Stationary distribution asymptotics in potential games

Combining the foregoing with minimum cost tree arguments yields:

**Proposition:** In a potential game with potential function $f$, we have

$$\lim_{N \to \infty} \lim_{\eta \to 0} \max_{x \in X^N} \left| -\frac{\eta}{N} \log \mu^{N,\eta}(x) - \Delta^+ f(x) \right| = 0,$$

where $\Delta^+ f(x) = \max_{i \in S} f(e_i) - f(x)$.

$\Rightarrow$ the “potential deficit” $\Delta^+ f(x)$ describes the rate of decay of $\mu^{N,\eta}(x)$ as $\eta \to 0$ when $N$ is large.

$\Rightarrow$ maximizers of $f$ are stochastically stable in the SNDL.

Remark: We do not need to assume that the size $N$ games are potential games, only that they converge to one as $N \to \infty$. Thus *exact* reversibility of $X^{N,\eta}$ is not used.
Example: A common interest game and its potential function

\[ F(x) = \begin{pmatrix} F_1(x) \\ F_2(x) \\ F_3(x) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}; \quad f(x) = \frac{1}{2} \left( (x_1)^2 + 2(x_2)^2 + 3(x_3)^2 \right). \]
Skewed games: optimal transition paths (1)
Skewed games: optimal transition paths (2)
Evaluating stochastic stability in skewed games: \[ A = \begin{pmatrix} 7 & 0 & 0 \\ 2 - q & 6 & 0 \\ 2 & 0 & 5 \end{pmatrix} \]
Evaluating stochastic stability in skewed games: \[ A = \begin{pmatrix} 7 & 0 & 0 \\ 2 - q & 6 & 0 \\ 2 & 0 & 5 \end{pmatrix} \]

\[ q \in [0, .3197) \]
Evaluating stochastic stability in skewed games: \[ A = \begin{pmatrix} 7 & 0 & 0 \\ 2 - q & 6 & 0 \\ 2 & 0 & 5 \end{pmatrix} \]

\[ q \in (0.3197, 1) \]
Evaluating stochastic stability in skewed games:  
\[
A = \begin{pmatrix}
7 & 0 & 0 \\
2 - q & 6 & 0 \\
2 & 0 & 5
\end{pmatrix}
\]

\[q \in (1, 3.4)\]
Evaluating stochastic stability in skewed games: \( A = \begin{pmatrix} 7 & 0 & 0 \\ 2 - q & 6 & 0 \\ 2 & 0 & 5 \end{pmatrix} \)

\[ q \in (3.4, 5) \]
The Large Population Limit

How does the process $X^{N, \eta}$ behave as $N$ grows large?
The Large Population Limit

How does the process $X_{N,\eta}^N$ behave as $N$ grows large?

Over finite time horizons, sample paths of $X_{N,\eta}^N$ can be approximated by deterministic trajectory: a solution to the mean dynamic ($M$).

The process $X_{N,\eta}^N$ will linger near the attractors of ($M$).

Over longer horizons, excursions against the flow of ($M$) can be described using the theory of sample path large deviations.
The analysis of the large population limit is much more difficult than that of the small noise limits.

The small noise limit concerns a sequence of stochastic processes on the fixed state space $X^N$.

We characterize large deviations in terms of solutions to deterministic control problems on $X^N$.

In the small noise double limit, the analysis of the second limit (in $N$) concerns a sequence of deterministic control problems on increasingly fine state spaces.

In contrast, the analysis of the large population limit considers a sequence of stochastic processes on increasingly fine state spaces.
The analysis of the large population limit is much more difficult than that of the small noise limits.

The small noise limit concerns a sequence of stochastic processes on the fixed state space $X^N$.

We characterize large deviations in terms of solutions to deterministic control problems on $X^N$.

In the small noise double limit, the analysis of the second limit (in $N$) concerns a sequence of deterministic control problems on increasingly fine state spaces.

In contrast, the analysis of the large population limit considers a sequence of stochastic processes on increasingly fine state spaces.
Finite horizon deterministic approximation

How does the process $X^{N,\eta}$ behave over time interval $[0, T]$ when $N$ is large?

The mean dynamic generated by $\sigma^{\eta}$ and $F$ is defined by the expected increments of $X^{N,\eta}$.

Let $\zeta^{\eta}_x$ be a random variable that describes normalized increment of $X^{N,\eta}$ from $x$.

The mean dynamic is

\[(M) \quad \dot{x} = V^{\eta}(x) \equiv \mathbb{E}\zeta^{\eta}_x.\]

Expressed componentwise:

$$\dot{x}_i = \sigma^{\eta}_i(F(x)) - x_i = \text{inflow into } i - \text{outflow from } i$$
Let $\hat{X}^{N,\eta} = \{\hat{X}^{N,\eta}_t\}_{t \geq 0}$ be the piecewise affine interpolation of $X^{N,\eta} = \{X^{N,\eta}_k\}_{k=0}^\infty$:

$$\hat{X}^{N,\eta}_t = X^{N,\eta}_{\lfloor Nt \rfloor} + (Nt - \lfloor Nt \rfloor)(X^{N,\eta}_{\lfloor Nt \rfloor+1} - X^{N,\eta}_{\lfloor Nt \rfloor}).$$

**Theorem:** (Kurtz (1970), Benaïm (1998), Benaïm and Weibull (2003))

Let $\{x_t\}_{t \geq 0}$ be the solution to (M) from $x_0 = \lim_{N \to \infty} X^N_0$.

Then for all $T < \infty$ and $\varepsilon > 0$,

$$\lim_{N \to \infty} \mathbb{P}\left( \sup_{t \in [0,T]} \left| \hat{X}^{N,\eta}_t - x_t \right| < \varepsilon \right) = 1.$$  

Convergence is exponential in $N$. 

Finite horizon deterministic approximation.
Example: The logit dynamic in a common interest game

\[
F(x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}; \quad \sigma_j^\eta(\pi) = \frac{\exp(\eta^{-1}\pi_j)}{\sum_{k \in S} \exp(\eta^{-1}\pi_k)}, \quad \eta = \frac{1}{4}.
\]
Sample path large deviations

The previous result showed that when \( N \) is large, \( X^{N,\eta} \) is likely to mirror a particular continuous path: a solution to (M).

The aim of the large deviations analysis is to assess the probabilities that paths unlike solutions of (M) are realized.

Difficulty: \( X^{N,\eta} \) runs on \( X^N \), so which paths are feasible varies with \( N \).

We assess the probability that the realization of the interpolated process \( \hat{X}^{N,\eta} \) lies in a given set of continuous paths \( \Phi \subset C[0, T] \).

\( (C[0, T] \) is the space of continuous paths through \( X \), endowed with the supremum norm.)
Large deviations analysis

Recall that $\zeta^\eta_x$ is the normalized increment of $X^{N,\eta}$ from $x$.

The Cramér transform of $\zeta^\eta_x$ is

$$L^\eta(x, z) = \sup_{u \in \mathbb{R}^n} \left( -\log \mathbb{E} e^{\langle u, \zeta^\eta_x - z \rangle} \right).$$

$L^\eta(x, z)$ is the “improbability” of motion from $x$ in direction $z$.

Intuition: Let $\bar{\zeta}^{N,\eta}_x$ be the sample mean of $N$ i.i.d. copies of $\zeta^\eta_x$.

**Theorem (Cramér):** Let $V \subset \mathbb{R}^n$. Then

$$\limsup_{N \to \infty} \frac{1}{N} \log \mathbb{P}(\bar{\zeta}^{N,\eta}_x \in V) \leq - \inf_{z \in \text{cl}(V)} L^\eta(x, z), \quad \text{and}$$

$$\liminf_{N \to \infty} \frac{1}{N} \log \mathbb{P}(\bar{\zeta}^{N,\eta}_x \in V) \geq - \inf_{z \in \text{int}(V)} L^\eta(x, z).$$

That is, $\mathbb{P}(\bar{\zeta}^{N,\eta}_x \in V) \approx \exp(-N L^\eta(x, z^*_V))$, where $z^*_V$ is “least unlikely” in $V$. 
Large deviations analysis

Recall that $\zeta^\eta_x$ is the normalized increment of $X^N,\eta$ from $x$.

The Cramér transform of $\zeta^\eta_x$ is

$$L^\eta(x, z) = \sup_{u \in \mathbb{R}^n} \left(-\log \mathbb{E} e^{\langle u, \zeta^\eta_x - z \rangle}\right).$$

$L^\eta(x, z)$ is the “improbability” of motion from $x$ in direction $z$.

Intuition: Let $\bar{\zeta}^N,\eta_x$ be the sample mean of $N$ i.i.d. copies of $\zeta^\eta_x$.

**Theorem (Cramér):** Let $V \subset \mathbb{R}^n$. Then

$$\limsup_{N \to \infty} \frac{1}{N} \log \mathbb{P}(\bar{\zeta}^N,\eta_x \in V) \leq - \inf_{z \in \text{cl}(V)} L^\eta(x, z), \text{ and}$$

$$\liminf_{N \to \infty} \frac{1}{N} \log \mathbb{P}(\bar{\zeta}^N,\eta_x \in V) \geq - \inf_{z \in \text{int}(V)} L^\eta(x, z).$$

That is, $\mathbb{P}(\bar{\zeta}^N,\eta_x \in V) \approx \exp(-N L^\eta(x, z^*_V))$, where $z^*_V$ is “least unlikely” in $V$. 
The path cost function $c_{x,T}(\cdot)$ adds up the improbabilities along path $\phi$:

$$c_T^n(\phi) = \int_0^T L^n(\phi_t, \dot{\phi}_t) \, dt.$$  

**Theorem:** Sample path large deviations principle

Suppose that the processes $\{X^{N,\eta}\}_{N=N_0}^\infty$ satisfy $\lim_{N \to \infty} X^{N,\eta}_0 = x \in X$. Let $\Phi \subset C[0,T]$ be a Borel set. Then

$$\limsup_{N \to \infty} \frac{1}{N} \log P(\hat{X}^{N,\eta}_{[0,T]} \in \Phi) \leq - \inf_{\phi \in \text{cl}(\Phi)} c_T^n(\phi), \quad \text{and}$$

$$\liminf_{N \to \infty} \frac{1}{N} \log P(\hat{X}^{N,\eta}_{[0,T]} \in \Phi) \geq - \inf_{\phi \in \text{int}(\Phi)} c_T^n(\phi).$$

In other words, $P(\hat{X}^{N,\eta}_{[0,T]} \in \Phi) \approx \exp(-Nc_T^n(\phi^*_\Phi))$. 
The path cost function \( c_{x,T}(\cdot) \) adds up the improbabilities along path \( \phi \):

\[
c_{T}^{\eta}(\phi) = \int_{0}^{T} L^{\eta}(\phi_t, \dot{\phi}_t) \, dt.
\]

**Theorem:** Sample path large deviations principle

*Suppose that the processes \( \{X^{N,\eta}\}_{N=N_0}^{\infty} \) satisfy \( \lim_{N \to \infty} X^{N,\eta}_0 = x \in X \).*

Let \( \Phi \subset C[0, T] \) be a Borel set. Then

\[
\limsup_{N \to \infty} \frac{1}{N} \log P(\hat{X}^{N,\eta}_{[0,T]} \in \Phi) \leq -\inf_{\phi \in \text{cl}(\Phi)} c_{T}^{\eta}(\phi), \quad \text{and}
\]

\[
\liminf_{N \to \infty} \frac{1}{N} \log P(\hat{X}^{N,\eta}_{[0,T]} \in \Phi) \geq -\inf_{\phi \in \text{int}(\Phi)} c_{T}^{\eta}(\phi).
\]

Versions of this result exist for processes on open domains
(Azencott and Ruget (1977), Dupuis (1988), Dupuis and Ellis (1997))

Our main contribution is to allow for processes
that run on a **compact** set—here the simplex \( X \).
Consequences

Combining the large deviation principle with methods of Freidlin and Wentzell (1984), we can, in the large population limit,

1. Describe the expected and likely time until exit from basins of attraction of stable equilibria.

2. Describe the asymptotic behavior of the stationary distribution, and thus the stochastically stable states

But these characterizations only allow us to obtain explicit solutions in special cases. . .
Summary and Next Steps

We considered stochastic evolutionary game dynamics based on noisy best response protocols.

We studied the rate of decay of the probabilities of transitions between attracting states.
Summary and Next Steps

We considered stochastic evolutionary game dynamics based on noisy best response protocols.

We studied the rate of decay of the probabilities of transitions between attracting states.

\[
\chi^{N,\eta} \\
\text{SNL} \\
\eta \to 0
\]

\[
\frac{1}{N} c^N(\phi^N) = \frac{1}{N} \sum_{k=0}^{i^N-1} L(\phi^N_k, \phi^N_k)
\]
Summary and Next Steps

We considered stochastic evolutionary game dynamics based on noisy best response protocols.

We studied the rate of decay of the probabilities of transitions between attracting states.

\[
\chi^{N,\eta} \\
N \to \infty \\
\eta \to 0 \\
Nc^N(\phi^N) = \frac{1}{N} \sum_{k=0}^{\ell^N-1} L(\phi_k^N, \phi_k^N) \\
(c(\phi) = \int_0^T L(\phi_t, \dot{\phi}_t) \, dt)
\]
Summary and Next Steps

We considered stochastic evolutionary game dynamics based on noisy best response protocols.

We studied the rate of decay of the probabilities of transitions between attracting states.

\[
\eta c^\eta(\phi) = \eta \int_0^T L^\eta(\phi_t, \dot{\phi}_t) \, dt
\]

\[
N \to \infty
\]

\[
\frac{1}{N} c^N(\phi^N) = \frac{1}{N} \sum_{k=0}^{N-1} L(\phi^N_k, \dot{\phi}^N_k)
\]

\[
N \to \infty
\]

\[
c(\phi) = \int_0^T L(\phi_t, \dot{\phi}_t) \, dt
\]
Summary and Next Steps

We considered stochastic evolutionary game dynamics based on noisy best response protocols.

We studied the rate of decay of the probabilities of transitions between attracting states.

\[
\frac{1}{N} c^N(\phi^N) = \frac{1}{N} \sum_{k=0}^{\ell^N-1} L(\phi_k^N, \phi_k^N) \xrightarrow[N \to \infty]{} \eta c^\eta(\phi) = \eta \int_0^T L^\eta(\phi_t, \dot{\phi}_t) \, dt
\]

\[
1 \xrightarrow[N \to \infty]{} \sum_{k=0}^{\ell^N-1} L(\phi_k^N, \phi_k^N) \xrightarrow[SNDL \eta \to 0]{} c(\phi) = \int_0^T L(\phi_t, \dot{\phi}_t) \, dt
\]
Summary and Next Steps

We considered stochastic evolutionary game dynamics based on noisy best response protocols.

We studied the rate of decay of the probabilities of transitions between attracting states.

\[ \frac{1}{N} c^N(\phi^N) = \frac{1}{N} \sum_{k=0}^{\ell^N - 1} L(\phi_k^N, \dot{\phi}_k^N) \]

**Proposition:** \( \lim_{\eta \to 0} \eta L^\eta(x, z) = L(x, z) \).

**Question:** Do \( \lim_{N \to \infty} \lim_{\eta \to 0} \mu^{N,\eta} \) and \( \lim_{\eta \to 0} \lim_{N \to \infty} \mu^{N,\eta} \) agree?