conditional variance that is much more precise than the best estimator of the conditional mean. For this reason and for simplicity we shall assume $V_{it} \equiv V_t$, is independent of i. It will be apparent as we illustrate our methods that this assumption can be relaxed at the cost of considerable complexity.

Assume, $w_i' = \rho_i(p' + y') + \varepsilon_i'$, and ε_i' conditionally independent of $p' + y'$, divide both sides of (4.2.1) by N, sum over i to obtain (conditional on the history of the economy at date t), suppressing t for ease of notation,

$$(1/N)\sum\{\tau_i E_i q' / V(q') - \rho_i\} \Rightarrow (1/V(q')\,[E^*\{\tau_i E_i q'\}] - E^*\rho_i\,,\ N \to \infty, \quad (4.2.2)$$

where E^* denotes expectation with respect to the measure, $\mu^*(A)$, defined by

$$(1/N)\sum I\,[(\tau_i\,, I_i\,, \rho_i) \in A] \Rightarrow \mu^*(A),\ N \to \infty. \quad (4.2.3)$$

Here I_i denotes the information set of trader i at date t, A is a set of agent characteristics (which includes choices), $I[(\tau_i\,, I_i\,, \rho_i) \in A$ is the indicator function of the event $[(\tau_i\,, I_i\,, \rho_i) \in A]$ which is unity if $(\tau_i\,, I_i\,, \rho_i) \in A$, zero otherwise, and $\Rightarrow$ denotes weak convergence. The theory of Section 4.1 locates sufficient conditions for the weak convergence of (4.2.3). We shall assume without further mention that these sufficient conditions hold.

Suppose there are x shares outstanding per trader. Then equilibration of demand and supply per trader yields, in the large economy limit, by (4.2.2),

$$(1/V(q')\,[E^*\{\tau_i E_i q'\}] - E^*\rho_i = x \quad (4.2.4)$$

We show the value of the modelling of Section 1 by applying it to a sequence of examples based on the above.

Example 4.2.1

Consider the "noise trader" theory of DeLong, Shleifer, Summers, and Waldman (1990), hereafter "DSSW." Let us use the theory of Section 4.1 to locate sufficient conditions for noise trader risk to matter in the large economy limit and to suggest a method of estimating the effect of noise traders using the methodology of Hansen and Singleton (1982).

For simplicity assume homogeneous conditional expectations on variance and an estimation procedure for the conditional mean with the following structure of errors across the set of noise traders, $\Omega = \{\text{bear, bull}\}^2 \equiv \{-1, +1\}^2$,

$$E_{it}(p' + y') = b_0\omega_{0it} + [1 + b_1\omega_{1it}]E_t(p' + y')\ , \quad (4.2.5)$$

where at each date t, $E_t(p' + y')$ is conditional expectation on a common information set available to all N traders, $\{\omega_{.it} \equiv (\omega_{0it}, \omega_{1it})\}$ is distributed according to a product form (like Example 4.2.3 below) of (4.1.3′), (4.1.5) where $u(\omega, t)$ is parameterized according to a measure of how well belief ω produced risk adjusted profits (utility) in the past.

In DSSW (1990) the bias in expectation is additive IID so $b_1 = 0$ captures the flavor of DSSW. So let us put $b_1 = 0$ for specificity. But, the reader should keep in mind that we can deal just as easily with multiplicative errors as additive errors. Put $x = \rho = 0$, assume constant risk tolerance across agents, bring back subscripts for clarity in (4.2.4) to obtain, from (4.1.19a,b),

$$Rp_t = b_0 m_t^* + E_t(p_{t+1} + y_{t+1}). \tag{4.2.6}$$

Write (4.2.6) in the form

$$E_t\{b_0 m_t^* + (p_{t+1} + y_{t+1}) - Rp_t\} = 0. \tag{4.2.7}$$

Equation (4.2.7) can be used to generate a set of orthogonality restrictions so that the parameters b_0, and the parameters embedded in m_t^* via (4.1.19) may be estimated (given a specification of behavior of $(\beta, J, h, u(.))$ over time) following the Generalized Instrumental Variables (GIV) used by Hansen and Singleton (1982). We speculate that the parameters of rather elaborate dynamic specifications could be estimated by adapting the simulation estimator methods of Hotz, Miller, Sanders, and Smith (1992). In this way returns data can speak to testing for the presence of noise traders with, for example, additive errors in formation of conditional expectations by testing $H_0 : b_0 = 0$ against the alternative $H_a : b_0$ not zero.

Of course some conditions must be imposed for the GIV procedure to "identify" the parameters of interest. A more serious problem with testing (4.2.7) concerns confusion of movements of the marginal rate of substitution in the CCAPM (Lucas (1978)) context tested by Hansen and Singleton with presence of noise traders in the context (4.2.7). But this problem could be dealt with by a noise trader component into the CCAPM setup of Hansen and Singleton (1982), following a procedure analogous to the above and deriving a general set of orthogonality conditions in which both the "pure" Hansen and Singleton CCAPM and the "pure" noise trader models are "nested."

Example 4.1 shows how a rich class of models may be formulated that *(i)* are econometrically tractable to GIV methodology, *(ii)* can be used to locate sufficient conditions for noise trader effects to survive the washing out effect of the law of large numbers, (There must be aggregate shocks to the $u(.\,, t)$ or

$\beta J > 1$), *(iii)* can be used to locate sufficient conditions for the additive IID errors of DSSW (1990) to appear in the large economy limit, *(iv)* can be enriched by different parameterizations of the $u(.\,,t)$ in (4.1.3′). We point out in passing that the presence of noise trading effects in the context (4.2.6) can be tested by using the West (1987) test. His procedure tests for the presence of terms like b_0 in linear present value models (4.2.6).

This is a good point to add a few words about justification for study of models with dispersion of beliefs. Antoniewicz (1992) in her work on volume reviews received work on volume dynamics. The consensus of this work is that trading volume is a very persistent series that is difficult to reduce to white noise by standard "detrending" methods.

Sargent (1992) shows how hard it is to preserve volume persistence in settings where the no-trade theorem becomes operative through learning. Therefore it appears that persistence in belief disparity will be needed if one is to get volume persistence out of belief disparity. While we shall exhibit models below that generate volume dynamics from heterogeneity in risk aversion and correlations of own income with the market these models do not seem right for explaining high frequency volume dynamics.

One justification for persistence in belief disparity is the work of Kurz (1990, 1991 and 1992) who develops a theory where all traders see the same data, form bulk quantities such as time averages, all time averages converge for each trader, yet disparity in limiting quantities remain. There is enough stationarity in Kurz's setting so that time averages converge, yet there is enough nonstationarity that each agent may not converge onto the same probability (the true probability). For the context of persistence of belief disparity it may be useful to think of Kurz's setting as a metaphor for a situation where data is arriving fast enough for each individual trader's estimators using time averages to converge but where the underlying system dynamics is changing slowly but fast enough that traders do not "lock onto" common agreement about the underlying probability. I.e. their estimators do not converge onto common limits.

Our type of modeling may have use in the future as a way of locating sufficient conditions on the degree of dependence of individual beliefs so that an aggregative effect remains in the cross sectional large economy limit. Kurz (1992) uses his theory to argue that the Dow was grossly overvalued in 1966. This argument requires that belief bias remain in the large economy limit. It is beyond the scope of our paper to say more about Kurz's stimulating work here. Suffice it to say that we believe that belief disparity plays an important role in volume dynamics and study of such models is justified. The dynamics of such models may be usefully disciplined by evolutionary modeling as in Blume and Easley (1992). Turn now to a related class of examples.

Example 4.2.2

Brock (1991a, p. 136-137) sketches a model where each trader has a choice of two strategies: -1 equals a chartist "trend chasing" strategy and $+1$ equals a "fundamentalist" strategy. Each of these strategies is a recipe for updating their estimate $E_{it}(p' + y')$ at each date t. Traders keep a record of the profits earned by the two strategies. Brock (1991a, p. 136-137) updated the "field" parameter h_t in (4.1.7) as a function of relative profits at t. We improve on this by using the theory in Section 4.1.

It is more natural to put $h_t = 0$ in (4.1.7) and define $u(\omega, t)$ to be the estimated profit for strategy $\omega \in \Omega \equiv \{-1, +1\}$, where the estimate is based upon the common information I_t available to traders at date t.

We define the fundamental strategy by putting

$$E_{+1,t}(p' + y') = E_t y' + E_t p'_{F,t+1}$$

where $\{p_{Ft}\}$ is the forward rational expectations solution process of the equation $Rp_t = E\{p_{t+1} + y_{t+1} \mid I_t\}$. As in Brock (1991a, p. 136), for strategy $\omega = -1$, put, $q_t \equiv p_t + y_t$,

$$E_{-1,t}\, q_{t+1} \equiv p_{Ft}/R + b_t,\ b_t \equiv b_{t-1} + \lambda(q_{t-1} - \mathrm{MA}(1, t-1)), \tag{4.2.8}$$

$$\mathrm{MA}(l, t-1) \equiv [q_{t-1} + \ldots + q_{t-1-(l-1)}]/l \equiv \text{moving average with } l \text{ lags.} \tag{4.2.9}$$

Suppose $\lambda > 0$. Note that $q_{t-1} > \mathrm{MA}(l, t-1)$ causes the bias over the fundamental to be increased; vice versa for "<".

Assume, for clarity that $\tau_i = \tau$, $\mathrm{Cov}_{it} \equiv 0$, $x = 0$. Close the model by using the expectation $E_{\omega_{it}}(q_{t+1})$, $\omega_{it} \in \{-1, +1\}$ to form the demands (4.2.1). Assume, at each date t, the probability trader i chooses ω_{it} is given by the MFT-discrete choice model (4.1.3′), (4.1.5).

We have a mixed discrete/continuous choice problem where (4.1.3′) serves as the discrete choice model for which strategy (conditional expectation) to use in forming demands. The continuous choice problem is the choice of optimum quantity of stock and bond to purchase given the conditional expectation (strategy). For each fixed date t, the $N \to \infty$ equilibrium (4.2.4) may be rewritten

$$Rp_t = [(1 - m_t^*)/2]E_{-1,t}(q_{t+1}) + [(1 + m_t^*)/2]E_{+1,t}(q_{1+t}), \tag{4.2.10}$$

where we choose m_t^* to be the largest (in absolute value) solution with the same sign as du to,

$$m = \{\exp[\beta Jm + \beta du] - \exp[-\beta Jm]\} / \{\exp[\beta Jm + \beta du] + \exp[-\beta J]\}$$

$$= \tanh[\beta(Jm + h')] \ , \qquad (4.2.11)$$

where $h' \equiv du/2$, $du \equiv u(+1, t) - u(-1, t)$ and $u(\omega, t)$ are measures of how well following strategy ω has generated utility for the trader had he followed it in the past. We assume this measure is a matter of public record available to all traders, but choice of ω is governed by (4.1.3′). One may now study the dynamics generated by (4.2.10). Unfortunately we must leave it to future research.

Example 4.2.3 (Based on Arthur, 1992)

Brian Arthur has written an interesting paper where he argues for replacing the deductive mode of theorizing by an inductive mode of theorizing. He shows that inductive modes are analytically tractable by considering a stock market where traders take positions by monitoring a collection of predictors $H_1, \ldots, H_P$. Suppose we encode these using bit strings $\omega \in \{-1, +1\}^L$ of length L as suggested at the end of Section 4.1. Introduce social interaction terms for each slot of the bit string and introduce a record for each predictor on how well it has done in the past. Base the utility $u(\omega, t)$ on this record at t. Let, at each date t, discrete choice occur according to the natural generalization of the discrete choice model (4.1.3′). Then join Arthur's approach and Example 4.2.2 to develop the dynamics. Our modification of Arthur allows "herding" which is induced by the interaction terms $\{J_l\}$.

The dynamics of this modified Arthur model should be very rich. It would be interesting to simulate it and see how easy it is to find parameters such that the output of returns and volume replicate the stylized facts reported by HTT which were discussed above. In principle the parameters of this modified Arthur model could be fitted to a subset of data to replicate relevant moments in sample. Then it could be evaluated by tests out-of-sample. Turn now to an example that generates trading volume via heterogeneity in correlations of own income with the market portfolio.

4.3. *A Model with Volume and Price Dynamics*

The volume dynamics are complicated in the general model (4.2.4), but they can be worked out and volume data may be used in estimation. However, simple volume dynamics may be obtained from (4.2.4) with

$$w_i' = \rho_i(p' + y') + \varepsilon_i', \qquad (4.3.1)$$

where $\{\rho_i\}$ has the probability structure (4.1.1), ε_i' is independent of p', y' and satisfies $(1/N)\sum\varepsilon_i' \Rightarrow 0$.

Assume there is supply of x shares per trader. Assuming homogeneous expectations on conditional mean and variance in (4.2.4), equating demand to supply of shares for N traders yields, introducing a first type of trader which has all ρ_i equal to a constant, we have,

$$x = (1/N)\sum D_i(p) = (1/N)\sum[\tau E_t q'/V_t(q')] - \rho_i]$$

$$= \tau E_t q'/V_t(q') - n_1\rho_1 - n_2\hat{\rho}_2, \qquad (4.3.2)$$

where

$$\hat{\rho}_2 \equiv \sum_i \rho_i/N_2 \Rightarrow \rho_2(m),\ N \to \infty, \qquad N_k = n_k N,\ n_k \quad \text{fixed},\ k = 1, 2\ . \qquad (4.3.3a)$$

Note that q' depends upon N but we abuse notation by neglecting this dependence in the notation. Here we suppose ρ_1 is constant across the N_1 type one traders, $\rho_2(\omega_{it})$ is the state of correlation for type two traders where $Pr\{\omega\}$ is given by (4.1.5).

This raises an issue of interpretation. One interpretation is to put $u(.) = 0$ and simply treat (4.1.5) as a convenient way to parsimoniously parameterize cross dependence of ρ in group two. Equation (4.1.5) may be motivated by placing the traders on a Durlauf (1991a,b) type lattice with probability structure (4.1.3) on the ρ's of (4.3.1). The lattice captures the relatedness of trader own incomes to each other. Equation (4.1.5) is an MFT approximation to (4.1.3) that is rough, but is accurate enough to suggest sufficient conditions for phase transition type behavior to take place (*cf.* Pearce, 1981). In any event this parametrization forces one to realize that *some* measure of cross dependence plays a key role in preventing the law of large numbers from "washing out" the ρ-effect, i.e, preventing $\hat{\rho}_2$ from converging to 0, as $N \to \infty$ unless this is "forced" by putting h not equal to zero. Small changes in h (or $u(.)$) can lead to large effects only when *some* measure of cross dependence is big enough. Equation (4.1.5) seems as attractive a way to capture this kind of effect as any.

Another interpretation is to imagine a discrete menu of funds with the same conditional variance but varying correlation with own income for group two traders. Consider the special case of a low correlation fund, -1 and a high correlation fund $+1$ Let a measure of past performance of each fund $u(\pm 1, t)$ be available at each date t. Then each member of group two picking which fund to buy shares in according to the discrete choice model (4.1.3) will lead to (4.1.5).

In this two state case, at each point in time, the limiting value of $\hat{\rho}_2$ will be

$$\rho_2(m) \equiv [(1-m)/2]\rho_2(-1) + [(1+m)/2]\rho_2(+1). \tag{4.3.3b}$$

Solve (4.3.2) for $E_t q' / V_t(q')$ to obtain

$$E_t q' / V_t(q') = [x + n_1\rho_1 + n_2\rho_2(m)] / t \equiv z_t \,. \tag{4.3.4}$$

In order to simplify the volume dynamics we ignore trading within group two and measure trading across groups one and two. Denote by D_{kt} the equilibrium demand by trader group $k = 1, 2$. With this qualification a natural measure of trading per capita per share can be generated from the following, which must hold in equilibrium,

$$D_{1t} - D_{1,t-1} = n_1\tau(z_t - z_{t-1}) \tag{4.3.5}$$

Motivated by (4.3.5) we define the turnover measure over the period $[t-1, t]$, denote it by V_t,

$$V_t \equiv n_1\tau(z_t - z_{t-1}) / x. \tag{4.3.6}$$

Equation (4.3.6) can be turned into a useful equation by parameterizing the volume dynamics via parameterization of $\{u_t(.), J_t, h_t\}$ as functions of, for example, past y-innovations and past volume. Given a probability structure on $\{y_t\}$, for example, Autoregressive with Independent and Identically Distributed (IID) or Martingale Difference Sequence (MDS) innovations, and a derived dynamics for $\{m_t\}$, where $m_t \equiv (J_t, h_t; u(.))$; equation (4.3.4) may be solved by forward iteration. This can be written as the conditional expectation of a capitalized sum of "adjusted" earnings where the capitalization factor is $1/R$. Both the price and volume dynamics can display abrupt changes to small changes in $u_t(.)$, h_t when $\beta J_t > 1$. We believe it would be interesting to "calibrate" models like Examples 4.1-4.3 and see how many of the stylized facts listed by HTT can be replicated. More will be said about this and other applications below.

4.4. *A Rational Expectations Models of Trading Volume and Liquidity Providers*

Campbell, Grossman, and Wang (1991) have developed a rational expectations model with two types of traders. Type A have constant risk aversion parameter a and type B have stochastic risk aversion parameter b_t at time t. We use the probability structure of Section 1 to "derive" a stochastic dynamics

for b_t. We outline how the model may be "solved" for a closed form solution by a dynamic variational approximation analysis.

We use similar notation as CGW. Put $R \equiv 1 + r$, $r > 0$ equal to return on the risk-free asset which is in perfectly elastic supply. Let $\bar{X}$ be supply per capita of stock, each share pays $D_t = D + d_t$, $d_t = \alpha d_{t-1} + u_t$, $0 \le \alpha \le 1$, u_t IID $(0, \sigma_u^2)$. There are two type of investors A, B with mean variance demands,

$$X_t^k = E[Q_{t+1} \mid I_t] / \psi_k \mathrm{Var}[Q_{t+1} \mid I_t],\ \psi_A \equiv a,\ \psi_B \equiv b_t,\ I_t = (P_t, D_t, S_t), \tag{4.4.1}$$

where $Q_{t+1} \equiv P_{t+1} + D_{t+1} - RP_t \equiv$ excess returns, $u_{t+1} = S_t + \varepsilon_{t+1}$, $\{S_t, \varepsilon_{t+1}\}$ is jointly IID with both means zero, $E[u_{t+1} \mid S_t] = S_t$, $\mathrm{Var}[u_{t+1} \mid S_t] = \sigma_\varepsilon^2$, $\mathrm{Var}[S_t] = \sigma_s^2$. Put

$$Z_t \equiv ab_t / [(1-\omega)a + \omega b_t],\quad \omega \equiv \text{fraction type } A, \tag{4.4.2}$$

assume $\{Z_t\}$ satisfies $E[Z_{t+1} \mid Z_t] = \gamma_0 + \gamma_1 Z_t$, $0 \le \gamma_1 \le 1$, $\mathrm{Var}[Z_{t+1} \mid Z_t] = \sigma_Z^2$, as sume $\sigma_Z^2 \le (R-\gamma_1)^2 (R-\alpha)^2 / 4\bar{X}^2 (R^2\sigma_\varepsilon^2 + \sigma_s^2)$. Then CGW show there is an equilibrium price function of the form,

$$P_t = p_0 + p_1 d_t + p_2 Z_t + p_3 S_t,\ p_1, p_3 > 0,\ p_2 < 0, \tag{4.4.3}$$

$$p_1 = \alpha / (R-\alpha),\ p_3 = 1/(R-\alpha),\ p_0 = (1/(R-1))\,[\bar{D} + \gamma_0 p_2], \tag{4.4.4}$$

$$p_2 = (1/(2\bar{X}\sigma_Z^2))\{-(R-\gamma_1) + [((R-\gamma_1)^2 - 4(1/(R-\alpha))^2(\bar{X}\sigma_Z^2)(R^2\sigma_\varepsilon^2 + \sigma_s^2)]^{1/2}\}, \tag{4.4.5}$$

$$Q_{t+1} = (\bar{D} - rp_0) + p_2[Z_{t+1} - RZ_t] + (1/(R-\alpha))S_{t+1} + (R/(R-\alpha))\varepsilon_{t+1}. \tag{4.4.6}$$

Add the demands, use the market clearing condition and the form of the solution price function to obtain

$$E[Q_{t+1} \mid I_t] = (\bar{X}\sigma_Q^2)Z_t,\ \mathrm{Var}(Q_{t+1} \mid I_t) \equiv \sigma_Q^2 = (1+p_1)^2\sigma_\varepsilon^2 + p_2^2\sigma_Z^2 + p_3^2\sigma_s^2. \tag{4.4.7}$$

Note that (4.4.7) says that excess returns are positive with the size increasing as the measure of average risk aversion, Z_t increases. Excess returns also increase as the conditional variance increases. However, note that conditional variance is constant. Hence the CGW model is not able to explain the well

known serial correlation structure of conditional variance, i.e. the Autoregressive Conditional Heteroscedasticity (ARCH) documented by the studies cited by Bollerslev, Chou, and Kroner (1992). This is because the CGW model is a linear model. Turn now to a nonlinear model which nests the CGW model.

Let there be three types of investors, A, B, C. Types A, B are as in CGW. At date t, member i of type C has risk tolerance given by $T_c(\omega_{it})$. Passing to the limit as the number of traders, N, goes to infinity but holding the fractions n_k, $k = A, B, C$ fixed we have, equating demand to supply,

$$E[Q_{t+1} \mid I_t] = (\bar{X}\sigma^2_{Qt})Z_t,\ \sigma^2_{Qt} \equiv \mathrm{Var}(Q_{t+1} \mid I_t),\ Z_t \equiv 1/[n_a\tau_a + n_b\tau_{bt} + n_c\tau_c(m_t)] \quad (4.4.8)$$

where

$$\tau_c(m) = [(1-m)/2]T_c(-1) + [(1+m)/2)]T_c(+1),$$

$m_t = m(J_t, h_t)$, $\tau_{bt} = 1/b_t$, $b_t \equiv$ risk aversion of type B as in CGW. If $\{(J_t, h_t)\}$ is a stochastic process such that $\{Z_t\}$ satisfied $E[Z_{t+1} \mid Z_t] = \gamma_0 + \gamma_1 Z_t$, $0 \le \gamma_1 \le 1$, $\mathrm{Var}[Z_{t+1} \mid Z_t] = \sigma^2_Z$ we could simply copy CGW and find their equilibrium price function.

But we want to parameterize $\{(J_t, h_t; u(.))\}$ as a function of past volume and past returns in such a way that we have the potential to replicate the stylized facts collected by HTT. This requires a nonconstant σ^2_Z and a natural way to introduce this is to parameterize J_t, h_t as functions of the past. For example, a large "aggregate dividend surprise", $D_t - E_{t-1}D_t$, may be associated with a change in the degree of dependence of risk tolerances in the future, i.e., a change in J_{t+1}.

While it is beyond the scope of this article to develop them, there are two routes to dealing with the third class of traders in the CGW model. The first one is to take a parameter like n and expand the equilibrium in a Taylor series in n_c around the value $n_c = 0$. In this way one can exploit the known CGW solution ($n_c = 0$) to build up an approximation to the unknown solution for positive n_c. The second route is to solve T period problems by backwards "dynamic programming" from a known terminal value p_T at T. A typical value for p_T is zero.

4.5. *An Asymmetric Information Rational Expectations Model*

Hellwig (1980) is a well known paper that derives a closed form solution for the large economy limit for a rational expectations model where N traders each receive signals about the future earnings of an asset. The solution shows how information is aggregated by the rational expectations price function in a competitive market.

Fix date t, suppress "t" in the notation, and append to Hellwig's model the following probability structure of signal quality across the set of N traders. If trader i is in state -1, let her signal variance be $S^2 > s^2$ which is her signal variance in state $+1$. Let $\omega \equiv (\omega_1, \ldots, \omega_N)$, $\omega_i \in \{-1, +1\}$ denote a configuration and let configuration probabilities be given by the Curie-Weiss probabilities treated in (4.1.17), (4.1.18) above. We have positioned ourselves to use Section 5 of Hellwig (1980) where he derives the form of the equilibrium price function in the large economy limit.

Define a trader to be "informed" if she is in state $+1$ so that her signal variance, s^2 is small. Traders in state "-1" are "uninformed". Now check that Hellwig's Assumptions B.1-B.4 are satisfied and take the large economy limit. Assume $(X, Z, \varepsilon_1, \ldots, \varepsilon_N)$ is Gaussian conditional on ω with the same diagonal variance covariance structure as Hellwig. Let f_-, f_+ denote the limiting fractions of uninformed and informed traders.

Look at Hellwig's equations (1980, p. 492), where we use his notation except we suppress the "upper *", write random variables as caps, put A equal to risk tolerance, and $B = A[f_- / S^2 + f_+ / s^2]$, where, by (4.1.17), (4.1.18),

$$f_- = (1 - m)/2,\ f_+ = (1 + m)/2,\ m = \theta(m) = \tanh(\beta J m + \beta h), \tag{4.5.1}$$

for $u(.) =$ constant,

$$P = \pi_0 + \pi X - \gamma Z, \tag{4.5.2}$$

$$\pi_0 \equiv [\bar{X}\Delta^2 A + \sigma^2 \bar{Z} A B] / D, \tag{4.5.3}$$

$$\pi \equiv [\sigma^2 B \Delta^2 + \sigma^2 A B^2] / D, \tag{4.5.4}$$

$$\gamma \equiv [\sigma^2 \Delta^2 + \sigma^2 A B] / D, \tag{4.5.5}$$

$$D \equiv \sigma^2 B \Delta^2 + \sigma^2 A B^2 + A \Delta^2 \tag{4.5.6}$$

Concentrate first on the case $u(.) =$ constant. If the mean field equation,

$$m = \tanh(\beta J m + \beta h), \tag{4.5.7}$$

has two solutions, choose the one with the same sign as h to be compatible with (4.1.19a).

The following four points may be made about this version of Hellwig's model. First, the correlatedness of the trader signal quality states may lead to

a "phase transition" where the equalibrium price relationship makes an abrupt shift in reponse to small changes in (J, h). Stephen Durlauf has made the important point that this kind of model can be used to show how large market movements may be caused by changes in the degree of correlation of information between agents rather than by large changes in the information itself.

Second, the model raises issues of how to measure factors that might effect the correlation strength of signal quality across agents. This in turn impacts on how rapidly the price function impounds information and impacts on the likelihood of abrupt changes in returns which may appear to be blowoffs and crashes.

Gennotte and Leland (1990) study how the sensitivity of demand of each trader type demands upon relative quality of signals and how this feeds into above changes in the price relationship provided their outside hedging function is upward sloping. The formula above shows how similar behavior can be obtained without the need for such an outside hedging function. Also note that it may be possible to "endogenize" the outside supply of shares, Z, by a community of noise traders modelled as in Section two above. A generalization of Hellwig (1980) to allow a probability structure on signals themselves, rather than just signal variances, like that in Section 1 would allow more abrupt changes in the level of prices to a small amount of "news", but that attempt must await future research.

Third, note the qualitative role of the correlation structure of signal receipts of inducing abrupt changes in the equilibrium price function, and, hence, in equilibrium returns. This feature is likely to remain in more elaborate models.

A fourth point is this. We may introduce a discrete choice decision into the model where we allow agents to choose high signal quality strategy, $\omega = +1$, (for which a fee of F is paid each period) or choose low signal quality strategy $\omega = -1$, (which is free). At each date t, choice is conducted according to the discrete choice model (4.1.3′) where $u(\omega, t)$ is based upon a measure of past performance of strategy choice $\omega \in \{-1, +1\}$. Two separate cases can be treated: *(i)* $u(\omega, t)$ is updated according to a publically kept record of experience with strategy ω; *(ii)* $u(\omega, t)$ is updated according to each individual trader's experience with ω. Discrete choice model (4.1.3), (4.1.3′) governs the probability structure in both cases.

A version of this model under research parameterizes correlation strength J as a function of past volume and past "surprises" at the time slot frequency. This is an attempt to capture the idea that high information channel congestion forces traders to condition on "coarse" information sets such as past prices which should lead to higher J which leads to higher volatility, i.e., larger changes in response to vibrations in h, $u(.)$. During periods of low

congestion traders should be able to get better quality signals on X from more independent sources so that J should be lower. Regardless of the loose heuristics, the idea is to parameterize J, h, $u(.)$ a functions of past price behavior, past volume, and past "surprises" (a measure of modulus of past forecast errors) in such a way that the data can speak to the form of this relationship. One version of this model that we have formulated leads to unpredictable first conditional moments of returns but somewhat predictable higher order conditional moments of returns.

The six applications above have been to financial models. We hasten to caution the reader that two period models and incomplete markets models, which we use to illustrate the usefulness of IPS methods are dangerous to apply in practice. This is partly because we have arbitrarily assumed that markets are incomplete in the Arrow-Debreu sense without giving a theory of why these markets are missing.

We have said nothing about the potentiality of options markets and other derivative security markets to ameliorate the potentiality for abrupt changes in returns in response to small events. Longer horizon models typically will lead to more smoothing behavior. More realistic models than those treated above will need to be investigated before it can be claimed that anything said in this paper pertains to financial reality. The point made in the financial section of this paper is simple: Models of this type are tractable to econometric methods such as Hansen and Singleton (1982), and Hotz *et al.* (1992). Indeed Tsibouris (1992) has estimated a version of an IPS model and tested the orthogonality restrictions with a degree of success comparable to received CCAPM theory. IPS models like those sketched above have the potential to help shed light on the puzzling stylized facts of HTT. Turn now to a very brief sketch how MFT/IPS/discrete choice methods may be useful in generating a new class of closed form solutions for simple macro/finance models.

4.6. *A Macro-Finance Equilibrium Asset Pricing Model with Interacting Agents*

We show off the flexibility of the approach to interactive systems modeling advertised above by exhibiting a macro-finance asset pricing model with a closed form solution. Consider Brock (1982, Example 1.5) where a representative "stand-in" consumer solves

$$\text{Max } E_0\{\sum_{t=0}^{t=\infty}\beta^{t-1}\log(c(t))\} \text{ s.t. } c_t + x_t = y_t \equiv \sum A_{it}x_{it-1}^{\alpha},\ \sum x_{it-1} \leq x_{t-1}, \quad (4.6.1)$$

where c_t, x_t, x_{it}, A_{it}, y_t, β, α, denote consumption, capital stock, capital stock allocated to process i, productivity shock to process i, total output plus total

capital stock carryover (all at date t), discount factor on future utility, and elasticity of production function. It is easy to see that the optimal solution of (4.6.1) is $x_t = \alpha\beta y_t$, $c_t = (1 - \alpha\beta)y_t$, $x_{it} = \eta_{it}x_t$, where the $\{\eta_{it}\}$ solve

$$\text{Max } E_t \log(\sum A_{i,t+1}\eta_{it}^{\alpha}x_t^{\alpha}), \text{ s.t. } \sum \eta_{it} = 1 \tag{4.6.2}$$

Note that (4.6.2) implies the $\{\eta_{it}\}$ do not depend upon x_t. We have now laid the foundation for building and solving an interacting systems model.

First, note that the solution form $x_t = \alpha\beta y_t$ does *not depend* upon the dynamic structure of $\{A_{it}\}$, hence we may preserve the same form of solution by introducing any pattern of externalities we wish and any number of agents we wish, so long as all of them are log utility maximizers facing problems with the same structure as (4.6.1), and all of them face the externalities parametrically when they solve their optimization problems. However, we wish to be able to compute statistics from aggregate quantities in order to make contact with Durlauf's (1991a, b) work on disparities among income and wealth across sites.

The solution for the $\{\eta_{it}\}$ in (4.6.2) is easy to find under the assumption that $Pr\{\omega_t\}$ is invariant to permutations within ω_t for each t. In this case we have

$$\nu_{it} = 1/N, \text{ for all } i, t, \tag{4.6.3}$$

$$x_t = \alpha\beta\sum A_{it}(1/N)^{\alpha}x_{t-1}^{\alpha}. \tag{4.6.4}$$

Given (4.6.3), (4.6.4) there are now two routes to obtaining a class of closed form solutions in the large economy limit, $N \to \infty$. First note that Section 1 locates sufficient conditions on the MFT/IPS probability structure for,

$$\sum A_{it}(1/N) \Rightarrow E^{*}A_{it}, \; N \to \infty, \tag{4.6.5}$$

so there is no problem for $\alpha = 1$. Second, in order to deal with $\alpha < 1$, consider an economy where $A_{it} = N^{1-\alpha}A_{0it}$. With this scaling (4.6.4) reduces to

$$x_t = \alpha\beta\sum A_{0it}(1/N)x_{t-1}^{\alpha}. \tag{4.6.6}$$

One may now investigate asset prices following Brock (1982) for specific examples such as simple MFT parameterizations of $A_{it} \equiv A(\omega_{it})$ with $\omega = -1$ for low A, $\omega = +1$ for high A using the simple equations (4.1.17), (4.1.18). In this way one can show how $\beta J > 1$, and an IID process for $\{h_t\}$ with mean zero and small variance can lead to big macro economic flucations.

A closely related type of example would be to replace the probability structure in Durlauf (1991a,b) with one of the MFT/IPS probability structures treated in this paper. The "Curie/Weiss" structure leading to (4.1.18) is simple enough to generate closed form solutions. The version of the discrete choice model reported by Proposition 4.1.2 is simple enough to apply to Durlauf's firms' choice of two technologies. While the resulting model would give something closer to a "closed form" solution, we doubt that it would be as rich as Durlauf's model.

5. Summary, Further Remarks, and Conclusions

This paper has tried to illustrate the usefulness of MFT/IPS methods as an input module into producing econometrically and analytically tractible models of use to finance and macroeconomics. We concentrated on finance and stressed the potentially of MFT/IPS models of addressing stylized facts which stress the apparent lack of connection of movement of stock returns and volume to "fundamentals". This is a natural place to argue for the promise of this type of model in being able to deal with stylized facts such as HTT (1991).

HTT (1991, p. 1006) state: "The large number of volatility shifts that we detect, and the fact that we are unable to find significant, real economic events in the neighborhood of a majority of these shifts, lead us to the conclusion that we may be observing instability in the noise component of volatility stemming from the microstructure of the stock market. Thus while our findings support the notion that changes in risk premia may serve to partially explain the excess volatility observed in stock prices, the apparently excessive volatility of volatility which we observe only serves to raise further questions regarding our ability to account fully for the behavior of stock prices through current financial markets paradigms".

Note that HTT stress the lack of a linkage between real economic events and the volatility shifts, and the asset pricing models sketched above generate large changes in response to small changes in du or h provided $\beta J > 1$. The parameter β is easy to interpret in the models built on the foundation of discrete choice such as (4.1.3). It is simply the intensity of choice and is a measure of the level of sharpness in choice. The parameter J is a measure of the strength of "ties" to a relevant "reference group" for each agent. Note that if intensity of choice is high we do not need much "sociology" for βJ to be greater than one. It is also plausible to think of parameterizing β, J as functions of the past history of the economy and estimating the parameters using, for example, the Generalized Instrumental Variable procedure (Hansen and Singleton, 1982). This is a good time to address a side issue that arises in IPS modeling.

IPS modeling is sometimes criticized in economics because it is said that there is no natural interpretation of the "inverse temperature" parameter β and even if there were the inverse temperature β is set exogenously such as controlling in a laboratory experiment or controlling by outside cooling or heating. Per Bak *et al.* (1992a, b) argue that sandpile models are superior to IPS models because the move to criticality is "self-organizing" rather than being forced exogenously.

While this argument has merit we believe that both types of models should be studied for the following reasons. *(i)* When IPS models are given a foundation in discrete choice random utility theory the interpretation of β becomes natural and we can imagine parameterizing it to capture economic incentives to make sharp or loose choices. *(ii)* The parameters J_{ij} become a tractable way to capture strong and weak ties between agents.

(iii) Since discrete choice econometric theory and IPS theory are well established we can draw on it to generate broad classes of econometrically tractable models as illustrated by the six examples above. Furthermore Anderson, de Palma, and Thisse (1993), show how there is a parallel between CES production functions and discrete choice theory and, hence, β is related to the elasticity of substitution in their CES production function. They show how welfare measures in discrete choice theory relate to production functions. The welfare measures treated in discrete choice theory are essentially the same as free energy expressions in IPS theory. This parallelism between economically interpretable quantities and physically interpretable quantities is beautiful and useful. *(iv)* Sandpile-based models still need an outside source (*e.g.* falling sand) to drive the pile to criticality. *(v)* The sandpile theory is not yet developed enough to conduct estimation and hypothesis testing which is fairly straightforward to do in the six examples laid out above. We conclude that it is wise to pursue both approaches because there are advantages and disadvantages to each.

Appendix

General Probability Structure with K Types of Interacting Agents

The interactions will be considered over disjoint sets $A_1, \ldots, A_K$ where types are homogeneous within each set but heterogeneous across each set. The large system limit (as $N \equiv$ total number $\to \infty$) will be taken by holding the fraction of each type $k = 1, 2 \ldots, K$ constant. To formalize this let Ω be a set of real numbers, let Ω_N be its N-fold Cartesian product, $\omega \in \Omega_N$,

$$Pr\{\omega\} = \exp[\beta G] P_N(\omega) / Z \ , \ G \equiv (1/2) \sum \sum M_k J_{kl}(N) M_l + \sum h_k M_k \ , \qquad (1)$$

where $M_k \equiv \sum \omega_i$ where $\sum$ is over i in A_k and $Z \equiv \sum \exp[G(\nu)]P_N(\nu)$ over all ν. Here $P_N(\nu)$ denotes the product probability on Ω_N induced by the common distribtion function F on Ω. We will concentrate on the case where Ω is finite and F is a sum of "dirac deltas" but use $\sum$ and $\int$ interchangeably to suggest the natural extension to a continuous state space. We shall also assume the utility functions $u(.)$ treated in Section 4 are constant. Once one sees how to generalize Section 4 for this case it will be straightforward to do it for utility functions.

The best way to think about this structure is to partition the vector ω thus: List first the components i in A_1, second the components i in A_2, etc. The probability structure captures homogeneous interactions within each set of entities $i \in A_k$ and captures heterogeneous interactions among entities across sets $A_1, \ldots, A_K$. The strength of interactions within A_k (across A_k, A_l) is measured by $J_{kk}(N)$ (by $J_{kl}(N)$) where the interaction strength will decrease linearly with N in this paper. That is to say the interaction strength becomes uniformly weaker across and within all sets of entities as N increases.

For future use, we want to find limiting values of the following statistics:

$$\hat{m}_k \equiv M_K / N_k \Rightarrow \langle \omega_i \rangle, \; i \in A_k, \tag{2}$$

where $N_k \equiv$ # of elements of A_k, $N_k / N = n_k$, and, N_k, $N \to \infty$ with n_k fixed. Here $\langle . \rangle$ denotes expectation with respect to the limiting probability, as $N \to \infty$, defined by (1) and $\Rightarrow$ denotes convergence in distribution. Details on how to define the object, $\langle . \rangle$, will follow in due course. We show now, that if we put $J_{kl}(N) = I_{kl} / N$, I_{kl} constant, the limiting value of (2) is given by a small generalization of Kac (1968).

At the risk of repeating material in the text, in order to see the Kac method with a minimum of clutter, deal first with the case $K = 1$, $I_{kl} \equiv J$, $h(A) = h$, $N_k = N$, $\hat{m}_k = \hat{m}$. Compute $Pr\{\omega\}$, $Z \equiv Z_N$. We have

$$Z_N \equiv \sum \exp \{ \beta[(J/2)(\sum \nu_i / N^{1/2})^2 + h(\sum \nu_i)]\} \; P_N(\nu) \tag{3}$$

$\sum$ is over $\nu \in \Omega_N$. Do the following steps. Put $\beta = 1$ to ease notation. First, use the identity

$$\exp[a^2] = (1/(2\pi))^{1/2} \int \exp[-x^2/2 + 2^{1/2} xa] dx, \tag{4}$$

and, second use the change of variable $y = x(J/N)^{1/2}$ to obtain

$$Pr\{\omega\} = (N/2\pi J)^{1/2} \int \exp[-y^2 N/2J] \prod \exp[(y+h)\omega_i] dy P_N(\omega) / Z_N, \tag{5}$$

$$Z_N \equiv (N/2\pi J)^{1/2}\int \exp[-y^2N/2J]\prod M[(y+h)]dy, \tag{6}$$

$$M(z) \equiv \sum_{\xi\in\Omega}\exp[z\xi]dF, \ \prod \text{ is product over } i = 1, 2, \ldots, N. \tag{7}$$

Note that we use M to denote "moment generating function" for (7). Compute

$$\begin{aligned} m &= \lim\{\langle [(1/N)(\textstyle\sum\omega_i)]\rangle\} \\ &= \lim\{\textstyle\int g(h+y)\,[K(y)]^N dy / \int K(\vartheta)^N d\vartheta\} \\ &\equiv \textstyle\int g(h+y)\mu_N(dy), \end{aligned} \tag{8}$$

where, $\mu_N(dy) \Rightarrow \delta_{y^*}(dy)$, $N \to \infty$,

$$K(y) \equiv M(h+y)\exp[-y^2/2J], \tag{9}$$

$$g(h+y) \equiv \int\{\xi\exp[\xi(h+y)]dF(\xi)\}/M(h+y) = M'(h+y)/M(h+y). \tag{10}$$

Apply Laplace's method (*cf.* Ellis, 1985) to see that, as $N \to \infty$, all probability mass is piled onto $y^* \equiv \text{Argmax}\{M(h+y)\exp[-y^2/J]\}$, i.e., $\mu_N(dy) \Rightarrow \delta_{y^*}(dy)$, $N \to \infty$. Hence,

$$y^* \text{ solves } JM'(h+y)/M(h+y) = y,\ m = M'(h+y^*)/M(h+y^*). \tag{11}$$

Now, Ellis (1985, p.38) shows $c(z) \equiv \log[M(z)]$ is convex, therefore $c'(z) = M'(z)/M(z)$ nondecreases in z. Make the modest additional assumption that $c'(z)$ increases in z. Then it is 1-1 and it follows that

$$m = c'(Jm+h) = M'(Jm+h)/M(Jm+h) \tag{12}$$

In order to study equations (11), (12) look at the special case, $\Omega \equiv \{-1, +1\}$, $dF(a) \equiv (1/2)\sum\delta_a$, where δ_a puts mass one on $a = -1, +1$, mass zero elsewhere. We have, recalling the definitions of hyperbolic cosine, sine, and tangent,

$$M(z) = \cosh(z),\ M'(z) = \sinh(z),\ c'(z) = \tanh(z), \tag{13}$$

$$m = \tanh(Jm+h) \tag{14}$$

Equation (14) is Ellis's Curie-Weiss mean field equation (Ellis, 1985, p. 180, p. 182) where we absorbed his β into J, h. Turn now to the discussion of this key equation.

Following Ellis it is easy to graph (14) and show that for $h = 0$, there is only one solution, $m = 0$; but, two solutions, $m_- = -m_+$, appear as soon as J becomes greater than one. For h not zero the one with the same sign as h is chosen. A "phase transition" or "spontaneous magnetisation" is said to appear when J becomes greater than one.

Before turning to central limit theorems, we remark that the solution properties outlined above can be generalized to the case where $dF(y) \equiv f(y)dy, f(-y) = f(y)$ and some regularity conditions. In this case one show $c'(-z) = -c'(z)$, $M'(0) = \int \xi dF = 0$, $M''(0) = \int \xi^2 dF$, $c'(0) = M''(0)$, so for $h = 0$ two solutions $m_- = -m_+$ appear for $JM''(0) > 1$, and $m = 0$ is the solution for $JM''(0) < 1$. Some conditions are needed on F to make $c'(z)$ display the qualitative properties of tanh which were used above.

Ellis (1985, pp. 187, 207, and reference to work of Ellis and Newman for general J, and h not zero) gives central limit theorems. In particular, for the case $J < 1$, $h = 0$ we have the central limit theorem

$$N^{1/2}(\hat{m} - m) \to N(0, \sigma^2(J, 0)),\ N \to \infty,\ \sigma^2(J, 0) \equiv (1 - J)^{-1}. \qquad (15)$$

Note how the variance tends to infinity as J tends to 1 from below.

Remark: It is easy to show using the same type of argument as that above that the covariances $\langle (\omega_i - m)(\omega_{i+L} - m) \rangle = 0$ in the limit for all integers L. That is why there are no covariance terms in (15). This appears to be a contradiction to the whole theme of this paper which is to show how models with correlated characteristics could be parsimoniously parameterized in such a way that econometric estimation is possible.

In order to explain this apparent contradiction we point out that Kac (1968, p. 258) shows that the Curie-Weiss probability structure we are using here is the limit as $\gamma \to 0$ of a class of structures indexed by γ which contain local interactions which *do give nonzero correlations*. As $\gamma \to 0$ the *range of interactions* becomes longer while the strength decreases in such a way that the Curie-Weiss equation (14) is obtained in the limit. In view of this "Kac bridge" between models with local strong interactions that have nonzero local correlations *whose strength increases with J* and the Curie-Weiss models with long range weak interactions that give the *same* equation (14) for the long run value of $\langle \omega_i \rangle$ we shall speak of an increase in J as an increase of local correlation of characteristics. Kac (1968) develops a series of expansions in γ for solutions for his general model where the Curie-Weiss theory appears as the lowest order of accuracy but accurate enough to display the phase

transition behavior that appears in the general model. In our view the analytical advantage of the Curie-Weiss structure and the Kac Bridge justifies the abuse of language we use in associating an increase in J with an increase in correlations across characteristics.

Turn now to the general case. We shall use an identity exploited by Kac (1968). In the applications below, inducing dynamics will give us flexible functional forms of dynamics on volume and stock returns, which will be one of our key applications. Another key application will be dynamics of K macro aggregates.

General Case: $K > 1$

Rewrite (1) as follows

$$Pr\{\omega\} = \exp[G]P_N(\omega)/Z,\ G \equiv (1/2)\sum\sum M_K J_{Kl}(N)M_l + \sum h_k M_k. \tag{16}$$

Put $N_j = n_j N$, $n_k^{1/2} J_{kl}(N) n_l^{1/2} \equiv n_k^{1/2} I_{kl} n_l^{1/2}/N \equiv J_{kl}/N$, J_{kl} constant,

$$G(\omega) \equiv (1/2)\sum\sum(M_k/N_k^{1/2})J_{kl}(M_l/N_l^{1/2}) + \sum h_k M_k. \tag{17}$$

Following Kac (1968, p. 254) use the following identity,

$$\exp\{(1/2)\sum\sum\xi_i A_{ij}\xi_j\} = (2\pi)^{-K/2}[\det(\mathbf{A})]^{-1/2}\int\exp[\sum\xi_i x_i - \mathbf{x}'\mathbf{A}^{-1}\mathbf{x}]/2d\mathbf{x}, \tag{18}$$

where $\sum$ is from 1 to K, bold face letters are vectors and matrices, $\int$ is over the K-vector $\mathbf{x}$, $\mathbf{A}$ is $K \times K$.
Put $A = J$, $C \equiv (2\pi)^{-K/2}[\det(\mathbf{A})]^{-1/2}$ and write

$$Pr\{\omega\} = CN[\prod n_j]^{1/2}\int\exp[\sum M_k(h_k + z_k) -$$

$$-(N/2)\sum\sum B_{kl}n_k n_l z_k z_l dz/Z$$

$$z_j \equiv y_j/N_j^{1/2} \tag{19}$$

after making a change of variable from y to z, letting the product $\prod$ run from $1, 2, \ldots, K$, and putting $B \equiv J^{-1}$. Application of Kac's identity and summing term by term allows one to show that z is given by

$$Z = CN[\prod n_j]^{1/2}\int\{\prod M(h_k + z_k)^{n_k}\exp[-(1/2)\sum\sum B_{kl}n_k n_l z_k z_l]\}^N dz, \tag{20}$$

We are now in a position to compute the limiting values, as $N \to \infty$, of moments. Consider

$$\langle M_j / N_j \rangle \text{ for set } A_j. \tag{21}$$

Use (19) and (20) to obtain

$$\langle M_j / N_j \rangle =$$

$$= \int [A(h_j + z_j) / M(h_j + z_j)] \{ \prod M(h_k + z_k)^{n_k} \exp[-\sum \sum B_{kl} n_k n_l z_k z_l] / 2 \}^N dz / Z \tag{22}$$

Here $A(y) \equiv \int \xi \exp[\xi y] dF(\xi) = M'(y)$. Use Laplace's method (Ellis, 1985) to observe that, as $N \to \infty$, all probability mass piles onto z^* where z^* maximizes

$$\sum n_k \log M(h_k + z_k) - (1/2) \sum \sum B_{kl} n_k n_l z_k z_l. \tag{23}$$

The first order necessary conditions for a maximum of (23) are given by

$$M'_k / M_k = \sum B_{kl} n_l z_l, \; M_k \equiv M(h_k + z_k). \tag{24}$$

Put $a_k \equiv M'_k / M_k$, $a \equiv (a_1, \ldots, a_K)$, $c_k \equiv n_k z_k$, $c \equiv (c_1, \ldots, c_K)$ and rewrite (24) thus,

$$a = Bc, \; Ja = c. \tag{25}$$

Recall that $J_{kl} = [n_k n_l]^{1/2} I_{kl}$, so (25) becomes

$$\sum_k [n_k n_l]^{1/2} I_{kl} M'_k / M_k = n_l z_l \;, \; l = 1, 2, \ldots, K. \tag{26}$$

Note that in the diagonal case $I_{kl} = 0$ for k not equal to l, and that n_l cancels from both sides of (26). In general the *relative size* $[n_k / n_l]^{1/2}$ plays a key role in transmitting interactions across different sets of entities as can be seen by dividing both sides of (26) by n_l.

We have

$$\hat{m}_k \equiv M_k / N_k \Rightarrow \langle \omega_i \rangle = M'(h_k + z_k^*) / M(h_k + z_k^*), \; i \in A_k. \tag{27}$$

Similar arguments yield, replacing $M_k \equiv \sum \omega_i$ by $\sum g(\omega_i)$ for any function g,

$$\sum g(\omega_i) / N_k \Rightarrow \int g(\xi) \exp[(h_k + z_k^*)\xi] dF(\xi) / M(h_k + z_k^*), \; i \in A_k. \tag{28}$$

These formulae for computation of limiting moments can be used to extend the applications given in the text.

Maximum Entropy and other Rationales

The probability structures put forth in Section 1 of our paper may appear arbitrary and chosen merely for convenience. There is some justification for the particular parameterization of probability structure that we chose to use. We give several arguments below. First we deal with the idea of modelling error-prone or "noise" traders. Then we show how such probabilities arise naturally from discrete choice theory.

A natural way to model the notion of "noisy beliefs" is to choose the most random probability measure subject to constraints. For example the most random probability measure on $\Omega \equiv \{-1, 1\}^N$ is the uniform measure that assigns $P(\omega) = 1/2^N$ to each $\omega \in \Omega$. Explanation of this idea requires a digression into the subject of maximum entropy measures.

Maximum Entropy Measures

To be precise consider the following optimization problem

$$\text{Maximize } [-\sum p(\omega)\ln(p(\omega))], \tag{29}$$

subject to,

$$\sum p(\omega)G(\omega) = G, \ \sum p(\omega) = 1, \tag{30}$$

where $\ln(x)$ denotes the natural logarithm of x, $\sum$ is over all $\omega \in \Omega$, and G denotes a fixed level of group sentiment. Let λ_1, λ_2 be the Lagrange multipliers associated with the two constraints in (29) by order of appearance. Then it is easy to show by differentiating the Lagrangian

$$L \equiv \sum -p(\omega)\ln[p(\omega)] + \lambda_1(G - \sum p(\omega)G(\omega)) + \lambda_2(1 - \sum p(\omega)), \tag{31}$$

that

$$p(\omega) = \exp[\beta G(\omega)]/Z; \ Z \equiv \sum_{\upsilon \in \Omega} \exp[\beta G(\upsilon)], \ \beta \equiv -\lambda_1. \tag{32}$$

Using the concavity of the function $H(x) \equiv -\ln(x)x$ on $(0, \infty)$ and the linearity of the two constraints in p, it is straightforward, using standard nonlinear programming theory, to show that β approaches $+\infty$ $(-\infty)$ as G approaches G^* (G_*) where G^* (G_*) denote the maximum (minimum) values

of G. Note that $p(\omega)$ collapses to the most uniform measure over Ω, i.e., the IID process over Ω, when $\beta = 0$. Denote this measure by π and note that $\pi(\omega) = 1/2^N$, for all $\omega \in \Omega$, and that (32) may be equivalently written by multiplying the numerator by $\pi(\omega)$ and each term of the denominator by $\pi(\upsilon)$. This is useful in Ellis's (1985) development of the limit theory which we follow. Also note that Ellis's β is absorbed in our J, h. To put it another way, Ellis's βJ, βh correspond to our J, h.

Rationale for Entropy Maximization

At this point we must further digress to discuss the rationale for entropy maximization. The motivation of entropy maximization stems from my own attempt to reformulate the "Harsanyi" doctrine or "common priors" assumption in such a way that some diversity of beliefs is allowed at a cost of a minimal number of free parameters.

The Harsanyi doctrine is controversial. Witness the labor expended defending it by Aumann against the flat statement by Kreps: "This assumption has very substantial implications for exchange among agents; we will encounter some of these later in the book. I leave it to others to defend this assumption –see, for example Aumann (1987, section 5)– as I cannot do so. But the reader should be alerted to this modeling assumption, which plays an important role in parts of modern microeconomic theory; it is called both the *common prior assumption* and the *Harsanyi doctrine*." (1990, p. 111). Kurz (1990), for example, makes a strong argument that diversity of beliefs will remain in the face of learning in a context where one would expect belief convergence.

In view of this conflict in the profession we propose a compromise. Entropy maximization subject to constraints is given a very spirited defense as a useful way to do prediction in statistical mechanics by E. T. Jaynes (1983) and there may be a useful analogy in economics as discussed by Zellner (1991). It may possibly be viewed as a way to allow some diversity in beliefs without emptying the theory of predictive content and in Bayesian literature as a way of giving some "objectivity" to "subjective" beliefs. I use it here to motivate an anaytically tractable model of interactive group formation of beliefs or sentiment. That is to say the group is assumed to have the most random set of group beliefs subject to a given mean level G. This restriction parsimoniously parameterizes the beliefs by three parameters (β, J, h) where β is fixed by G.

A very innovative use of entropy and the methodology of Gibbsian statistical mechanics is in Stutzer's work (*cf.* Stutzer (1992) and references to his earlier papers). He uses this methodology to put forth a concept of financial entropy which he relates to the degree of risk adjustment required

of any arbitrage-free asset pricing theory to explain the risk premia of a given set of assets. He applies his theory to data on the stock and bond markets and produces evidence consistent with a secular decline in the influence of risk aversion in the stock and bond markets over the past 65 years. We urge the reader to study Stutzer's work.

If the reader does not care for the maximum entropy argument the same probabilities may be derived, as in Section 4.1, by viewing the group of interactive noise traders as solving the "social discrete stochastic choice problem"

$$\underset{\omega \in \Omega}{\text{Maximize}}\ G(\omega) + \mu e(\omega),\ \beta \equiv \mu^{-1} \tag{33}$$

where $\{e(\omega)\}$ is IID extreme value distributed. It is pointed out in Manski and McFadden (1980) that Prob{ choose ω} is exactly equal to the logit probability (32). Since the probabilities are logit we have access to the extensive econometric literature on estimation of logit systems. Indeed this is a main part of the motivation for the type of theory we are building. More will be said about estimation in future work.

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[1] Not all of the references are cited in the text. Since part of the purpose of this paper is to provide a guide to the literature, I have listed many references here. The Brock, Hsieh, LeBaron book cited below many references which are not cited here.

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