Some examples of NPV and IRR

The number of examples are legion, especially when one realizes that loans have the opposite sign values of the cash flows than investment projects. In a loan, we receive money in period 0 (today) and make payments in the future.

I am in the market to buy another car. My current car is worth $3,000 and the car I wish to purchase is $25,000. I can put down $5,000 and have to borrow the rest at rate 6.8% for thirty six months.

**Draw picture of cash flows** Positive inflow at time 0 of $17,000 and monthly outflows of payments \((pmt)\). To determine the size of the monthly payments, \(x\), solve for \(x\):

\[
17,000 - \sum_{i=1}^{3 \cdot 12} \frac{x}{(1 + \frac{0.068}{12})^i} = 0
\]

The term in the denominator reflects that interest is compounded monthly (applied to the outstanding balance of the loan) while the interest rate is quoted as an annual rate.

My fancy calculator tells me the monthly payments will be 523 dollars per month (and some change). If you don’t have a fancy calculator we can solve the equation directly.

Let \(n\) denote the length of the loan in months \((3 \times 12)\) and \(q = r/m\) to be the effective interest rate per month. Let \(PV\) equal the loan amount \((PV\) reminds us that the loan comes in today and payments are in the future). Write the equation as:

\[
PV - \sum_{i=1}^{n} \frac{x}{(1 + q)^i} = 0
\]

Notice that \(\frac{1}{1+q} \leq 1\) as \(q \geq 0\). So the term \(1/(1 + q)^j\) declines as \(j\) increases, which reflects that payments made farther into the future are smaller in today’s dollars.

Because the monthly payment is constant it can be moved outside the summation sign. Let \(u = 1/(1 + q)\), and the equation becomes:

\[
PV = x \left[ \sum_{j=1}^{n} u^j \right]
\]

\[
= x B_n(u), \quad B_n(u) = \sum_{j=1}^{n} u^j
\]

\[
x = \frac{PV}{B_n(u)}
\]
So to find $x$, we need to only find an expression for $B_n(u)$.

Multiply $B_n(u)$ by $u$ and subtract it from $B_n(u)$ to get

\[
\begin{align*}
B_n(u) &= u + u^2 + u^3 + \cdots + u^n \\
u B_n(u) &= u^2 + u^3 + \cdots + u^{n+1} \\
B_n(u) - u B_n(u) &= u - u^{n+1} = u(1 - u^n) \\
(1 - u)B_n(u) &= u(1 - u^n) \\
B_n(u) &= \frac{u(1 - u^n)}{1 - u}
\end{align*}
\]

Substitute for $u = 1/(1 + q)$ to yield

\[
x = \frac{q \ PV}{1 - (1 + q)^{-n}}
\]

Substituting in

\[
x = \frac{.00567 \cdot 17,000}{1 - 1.00567^{-36}} \\
= \frac{96.39}{1 - .81583} = \frac{96.39}{.18417} \\
= 523.39
\]

**Continuous Time Flows**

Recall that interest compounded $n$ times per year such as monthly or quarterly for $t$ years accumulates as $a(t)$

\[
a(t) = \left(1 + \frac{r}{n}\right)^{nt}
\]

Continuous compounding is then the compounding period is infinitely small, thus $n \to \infty$.

\[
a(t) = \lim_{n \to \infty} \left(1 + \frac{r}{n}\right)^{nt} = e^{rt}
\]

So, an initial amount, $A_0$ compounded continuously at interest rate $r$ for $t$ periods (where the time units on $r$ matches that of $t$) is

\[
A(t) = A_0a(t) = A_0e^{rt}.
\]
High School versus College

Consider a person who just graduated from high school. They have an offer of employment that pays them a salary of \( Y_{hs}(t) \) (per unit of time \( t \)). They need not make any investment and will work at this establishment until retirement. (I realize assuming no job change is unrealistic, but notice that to permit turnover and job change requires that we state when it will occur and its payoff. So for now, let’s abstract from such complications.)

Quantities are in real terms, and the assumption is that the high school job pays \( Y_{hs}(t) = Y_{hs} \) until age \( T \) (say 65). The market interest rate is \( r \), assumed constant and the Present value is:

\[
PV_{hs} = \int_0^T Y_{hs} e^{-rt} dt
\]

\[
= \left. \frac{-Y_{hs}}{r} \right|_0^T
\]

\[
= \frac{-Y_{hs}}{r} [e^{-rT} - 1]
\]

\[
= \frac{Y_{hs}}{r} \left[ 1 - e^{-rT} \right].
\]

But the power of discounting is such that for reasonable sized values of \( T \) and for \( r, e^{-rT} \approx 0 \). So the present value of holding the job forever is

\[
Y_{hs}/r
\]

but this is just the price one should be willing to pay for an asset that pays \( Y_{hs} \) per year in perpetuity.

Now consider the cost of going to college. The (full) flow cost of college is \( C_t = C \) for \( s \) periods (e.g., 4 years or 5 years or . . .). Again let the real interest rate be denoted as \( r \) and the earnings of a college graduate be \( Y_c(t) = Y_c \).

The direct benefit is

\[
PV_c = -\int_0^s C e^{-rt} dt + \int_s^\infty Y_c e^{-rt} dt
\]

\[
= \frac{-C}{r} [1 - e^{-rs}] + \frac{Y_c}{r} e^{-rs}
\]

\[
= \frac{(Y_c + C)e^{-rs}}{r} - \frac{C}{r}
\]

Substituting the PV of High School as the opportunity cost of going to college, the NPV of obtaining a college degree is
\[
NPV_c = \frac{(Y_c + C)e^{-rs}}{r} - \frac{C}{r} - \frac{Y_{hs}}{r}
\]

Can now investigate comparative statics of the effect of differential earnings and out-of-pocket costs and financial aid. Should write \(NPV_c\) as \(NPV_c(s, Y_c, C, Y_{hs}, r)\), but to keep things simple do not.

The \(NPV_c\) increasing in \(Y_c\), and decreasing in \(C\) and \(Y_{hs}\), and in \(s\) (the longer it takes to go through college the lower the net present value of a college education.

What is the increased cost of taking \(s + 1\) years versus \(s\) years? At the start of the college education,

\[
NPV_c(s) - NPV_c(s + 1) = r^{-1}(Y_c + C)(e^{-rs} - e^{-r(s+1)})
\]

which is approximately (using calculus)

\[
\frac{d NPV_c(s)}{ds} \approx (Y_c + C)e^{-rs}
\]

So the increased cost is the additional year of cost plus the forgone income as a college graduate. But the discount factor discounts back to time 0 the start of the investment process. If you decide in the spring of what was to be the last semester, in contemporary dollars, postponing only one year, so the appropriate discount factor is \(e^{-r}\) not \(e^{-rs}\). So, if direct costs are $15,000 and earnings as a college graduate are $30,000, and the interest rate is 3%, the incremental cost of an additional year is $45,000 × 0.97 = $43,670. Enjoy.

What is the effect of an increase in the real interest rate?

This is the basic model for understanding enrollments in college. Predictions follow from the comparative statics mentioned above.

1. College enrollment will decline if costs of college rise (holding other factors constant)
2. College enrollment will increase if the gap between earnings of college graduates and high school graduates widens.
3. Most college students will be young. Why?
4. The cost of college increases with the opportunity of going to school. Thus, most students will be young before their value in the market increases and the opportunity cost is too large to make the investment.

There is a second reason for why students are likely to be young, and that is that the investment return increases with the length of the payoff period. I assumed the payoff period was sufficiently long \(e^{-rT}\) so that we could consider the flows inconsequential \((\approx 0)\). Streams over relatively short duration of 5 or 10 years are not close to zero.

Thus, have the incentive to invest in college education when 18 because the opportunity cost of your time is low, and you have the longest period to reap the benefits.
Human capital theory is a common way to understand the distribution of earnings. Earnings across a wide variety of societies have three common features:

1. Average earnings of full-time workers rise with the level of education.

2. The most rapid increase in earnings occurs early in one’s working life, thus giving a concave shape to the age–earnings profile of both men and women;

3. Age–earnings profiles tend to fan out, so that education–related earnings differences later in workers’ lives are greater than those early on;

**Draw standard concave age–earnings profile.**

The first one is obvious and indeed necessary: people with more education foregone consumption while in school, and must be paid more later to recover their investment. Indeed, we could imagine an economy in which people are identical, same preferences, same productive abilities. There can be two jobs, one that requires no schooling and another that requires some schooling. Since people are identical they are indifferent which job they do, so their utility has to be the same. So, induce some people to accept the job that require some education, earnings in that job has to be higher than in the job without education. Workers in the job with education are not better off, just paid more when older and receive less when young.

Strong earnings growth occurs when young? Think of a production function, in which human capital depends on schooling and experience. (Human capital is the acquisition of knowledge and abilities, learn some things in school others on the job.) In most production function, the marginal returns are largest when input levels are low.

Why do earnings fan out? **Ask.** People who are more able or can learn more quickly will accumulate knowledge faster. And the fast learners are likely to be the ones that acquired the most schooling.

And it is true that more educated workers are less likely to be unemployed. If we think of learning–by–doing investment process while working, unemployed workers accumulate less human capital.

**Private Versus Social Returns**

Increased earnings due to education (and the investment in human capital) or increased earnings because of moving from a poor to good labor market are examples of private returns.