Information Aggregation in Common Value Asset Markets and the Efficient Markets Hypothesis*

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Abstract

This paper studies information aggregation in common value double auctions with a continuum of traders. The population includes both sophisticated and naïve traders. Existence and uniqueness of monotone equilibrium prices is shown under mild conditions on the distribution of naïve bids. The set of possible asset values exhibits two distinct regions: a region where prices equal values, and a mispricing region featuring a *local* favorite-longshot bias. If the proportion of naïve traders falls below a (strictly positive) lower bound, prices are always correct. In contrast, when the presence of naïve traders is above some upper bound there is mispricing almost everywhere. This indicates that the efficient markets hypothesis can hold when there exist non-negligible levels of noise trade, although even a moderate presence of boundedly rational traders can lead to severe mispricing. It is shown through an example that, when there is uncertainty about the fraction of naïve traders, the same qualitative results hold in expectation. An empirical method to identify the mispricing region is suggested.

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1 Introduction

Markets have long been touted not only for their role in allocating goods, but also for their ability to efficiently process information about the objects of trade. The *efficient markets hypothesis* (Fama (1970)) postulates that prices in competitive markets "fully reflect" all the available information, which

> ... never exists in concentrated or integrated form, but solely as the dispersed bits of incomplete and frequently contradictory knowledge which all the separate individuals possess.
>
> Hayek (1945, p. 519).

Based on this conjecture, exchange institutions designed with the sole purpose of forecasting future events, commonly referred to as *prediction markets*, have emerged in the last two decades.[1] Although empirical evidence suggests that market prices seem to perform well as information aggregators,[2] it is not clear how prices behave in the presence of boundedly rational agents. Experimental evidence points to the existence of two basic types of traders in these markets: experienced *arbitrageurs* who seem to have a good understanding of the markets and do not exhibit behavioral biases, and naïve traders who systematically earn lower profits and follow seemingly suboptimal trading strategies.[3] In addition, there is extensive empirical evidence pointing to pervasive abnormal (i.e. risk adjusted) average returns in financial markets.[4]

The goal of this paper is to analyze whether prices accurately reflect asset values in auction-based asset markets when the trader population consists of both perfectly rational (*sophisticated*) and boundedly rational (*naïve*) traders. More specifically, I

[1]Examples of prediction markets include the Iowa Electronic Markets (IEM) for presidential elections, the Hewlett-Packard internal market to predict future sales and Hollywood exchange – a virtual currency market aimed at forecasting movie ticket sales. I refer the reader to Wolfers and Zitzewitz (2004) for a more comprehensive list of existing markets.

[2]See, for instance, Forsythe, Nelson, Neumann, and Wright (1992), Berg, Forsythe, Nelson, and Rietz (2005), or Berg, Nelson, and Rietz (2003) for evidence from the IEM, and Chen and Plott (2002) for a study of the internal market at Hewlett-Packard.

[3]Using data from the IEM 1988 presidential election market, Forsythe, Nelson, Neumann, and Wright (1992) find evidence of this categorization: some traders exhibited ideological biases (i.e. their party affiliation influenced their trading behavior) and earned negative average returns and there were also what they called "marginal traders" who did not exhibit such biases and earned a 10% average return. They argue that the presence of the latter leads to information aggregation, despite the pool of traders in the market not being a representative sample of the voting population.

[4]See for instance the recent analysis of Fama and French (2008). Also, Hirshleifer (2001) and Barberis and Thaler (2006) provide detailed surveys of these anomalies and argue that such phenomena can be better understood by relaxing the assumption that all agents are perfectly rational.

study price formation in markets by modeling them as common value double auctions (CVDA) in which risk-neutral traders receive a private signal stochastically related to the value of the security traded. The reasons behind this choice are threefold. First, the double auction mechanism resembles existing markets, in which both buyers and sellers post limit orders to respectively buy and sell units of the asset. Second, in a common value setting information aggregation refers to the process of aggregating traders' private signals about the unknown value of the asset, and thus traders can infer something about the value from observing prices.[5] Finally, if the efficient markets hypothesis (EMH) holds, pure common values plus risk neutrality imply that security prices in prediction markets can be directly interpreted as unbiased estimates of some parameter of the probability distribution of the events to be forecasted.

By having a heterogeneous trader papulation I can look at the extent of mispricing and its qualitative features as a function of the fraction of naïve traders in the population. To this end, I provide a very general definition of naïve bidding behavior, which includes both the noise traders used in rational expectations equilibrium (REE) and market microstructure models and many of the behavioral types recently proposed in the auction literature (e.g. *fully cursed* traders, *level-k* thinking).

A known issue in double auctions is that traders' ability to affect prices in finite agent environments makes equilibrium analysis quite intractable. However, since individuals lose their market power as the market grows, I look at a limit case with a continuum of agents.

I show that when the distribution of naive bids satisfies a mild condition, equilibria exist in which the price is monotone with respect to the asset value; and that while multiple such equilibria may exist, the equilibrium mapping from values to prices is essentially unique. Furthermore, I show that in any monotone equilibrium sophisticated traders place their bids either in regions of the bidding space where prices are equal to asset values or outside the range of prices. Accordingly, prices are characterized by having their range partitioned into two distinct regions: a region where prices equal asset values and a mispricing area where prices differ from values and are completely determined by naïve bids. Furthermore, mispricing takes on a very specific shape, which I call *local* favorite-longshot bias: low values are always overpriced and high values are always underpriced in intervals of asset values where there is mispricing. I also show using an example that these results continue to hold in expectation when there is aggregate uncertainty about the fraction of naïve traders in the market: the mapping from expected asset values (conditional on prices) to

[5] Recent models of prediction markets proposed by Manski (2006), Gjerstad (2005) and Wolfers and Zitzewitz (2006) assume that agents have pure private values (referred to as beliefs). Each agent knows exactly her valuation of the asset and thus prices do not provide relevant information to traders. On the other hand, Ottaviani and Sørensen (2007) propose an REE model in which traders have heterogeneous priors and they update their beliefs after observing the market price, which is perfectly revealing by assumption.

market prices has two distinct regions, one exhibiting (expected) mispricing and the other in which prices equal conditional expected values.[6]

There are three distinct scenarios: when the fraction of naïve traders is small, prices equal asset values everywhere. When there is a moderate presence of naïve bidders, the equilibrium price function has both a mispricing and an accurate pricing region. Finally, if the share of naïve traders surpasses some threshold they always determine the price, leading to mispricing almost everywhere. This result implies that a non-negligible presence of boundedly rational traders does not necessarily lead to mispricing. However, there can be substantial mispricing for moderate fractions of naïve traders, which is at odds with the idea that a small presence of sophisticated traders suffices to get full information aggregation. In some sense, this is both good and bad news for the EMH and applications based on it such as prediction markets: the EMH is robust to small "perturbations" in an otherwise homogeneous trader population, but it will not likely hold in markets with a not so small presence of irrational or unexperienced traders.

It is worth mentioning that having risk-averse or risk-loving sophisticated traders would lead to qualitatively similar prices. This is so because the key feature of their bidding behavior, i.e. that they avoid placing bids in regions of the bidding space where prices differ from values, does not depend on their risk attitudes. Hence, the forecasting properties of prediction markets may not heavily depend on eliciting risk neutrality.

Based on this result, mispricing can be empirically identified relying solely on price data and density of bids. The density of bids in a region without mispricing will generally be higher than the density in a mispricing region.

This paper is organized as follows. First I describe a typical prediction market, highlighting its relevant features. I then look at existing theories that address information aggregation through the price mechanism. The common value double auction model is laid out in section two. Section three provides the characterization of equilibrium prices in a continuum agent economy. An empirical test of information aggregation is then suggested. Before concluding, I briefly discuss the possibility of relaxing some of the assumptions of the model.

1.1 Prediction Markets: Prices as Forecasting Devices

Assume our goal is to predict the outcome of a U.S. presidential election, to be held at time T. There are two candidates, Barak and John. We would like to forecast at any time $t < T$ the vote share each candidate will get in the election. Denote the respective vote shares by v^B and $v^J = 1 - v^B$, respectively. Each individual agent has some information about v^B.

[6] With aggregate uncertainty prices are no longer measurable with respect to asset values, i.e. they are not fully revealing.

For this purpose, we set up a futures market in which agents can post and accept offers to trade two futures contracts: the "Barak security" (denoted by B), which pays, at time T, ¢1 per Barak's percentage point of the popular vote; and the "John security" (J), which pays ¢1 per John's percentage point of the popular vote. According to the EMH, the price of each security at any time t reflects its expected *return* conditional on the pool of *all* relevant information individual traders may have at t. The value of the expected return is a function of its liquidation value v^j ($j = B, J$) and its risk.[7] Assuming agents exhibit risk neutrality and share common values then $p_t^j = \mathbb{E}(v^j | \mathscr{F}_t)$, where p_t^j is the price at time t of security j and $\mathscr{F}_t$ represents all the information available at t. Thus, if the EMH is correct and traders are money-maximizers with no insurance motives, security prices provide the "best" vote share estimates, given the information available at that period.

In practice, prediction markets are futures markets with a distinct feature: while standard futures markets perform the functions of price discovery and risk transfer, the latter is absent from prediction markets, i.e. their sole purpose is to aggregate information and produce forecasts through prices. In fact, they typically impose tight investment limits (and a ban on short sales) to induce price taking and to ensure that traders' participation is not driven by hedging.[8]

Apart from these restrictions, trading rules resemble those of existing stock exchanges (without market dealers). In essence, they are continuous-time double auctions: traders can either post limit orders to sell or buy each of the securities (known as *asks* and *bids*, respectively) or accept outstanding offers. An ask (bid) specifies the maximum number of units of the security to sell (buy) and the minimum (maximum) price to be accepted. The two outstanding offers are the ask with the lowest price and the bid with the highest price. At each moment, traders are informed of the outstanding offers and the price of the last transaction.[9]

The empirical evidence generally suggests that prices perform well as forecasts,[10] although Cowgill, Wolfers, and Zitzewitz (2009) find evidence of mispricing in some internal prediction markets run at Google, which they attribute to the presence of unexperienced employees. In addition, experimental evidence in common value double auctions suggests that prices aggregate information, providing support for the EMH (Plott and Sunder (1988), Forsythe and Lundholm (1990), Guarnaschelli, Kwasnica, and Plott (2003)).

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[7] Since asset pricing models differ in the definition of risk, the EMH is a joint hypothesis.

[8] For instance, the investment limit at the IEM is $500.

[9] Information on trade volume, price and bid/ask spread histories is also available.

[10] For instance, Forsythe, Nelson, Neumann, and Wright (1992), Berg, Forsythe, Nelson, and Rietz (2005) and Berg, Nelson, and Rietz (2003) find that market prices were consistently closer than opinion polls to actual vote shares at the IEM. Chen and Plott (2002) show that prices in the internal market set up by Hewlett-Packard were closer to the company sales than official forecasts.

1.2 Related literature

This paper is related to two main strands of literature. The first deals with markets in which both sophisticated traders and noise traders are present while the second deals with prices in common value auctions. The main focus of existing models with noise traders is to understand how the risk introduced by noise traders affects market prices. In these models, risk aversion on the side of sophisticated traders imposes and endogenous limit on their demand for the asset. In noisy REE models such as Hellwig (1980) and the strategic version of Kyle (1989), prices are not fully revealing because they incorporate residual uncertainty about the aggregate noise trader demand due to sophisticated traders' limited positions. In their model, noise trader demand is completely insensitive to prices and this leads to no expected mispricing in the limit (as the number of traders grows to infinity). However, De Long, Shleifer, Summers, and Waldmann (1990) show using an REE overlapping generations model that mispricing can be substantial when noise trader demand is a function of prices, even in the absence of fundamental risk.[11] Moreover, they show that noise traders can earn higher expected profits than sophisticated traders.

My paper departs from this literature in several ways. First, my focus is solely on speculative trade, i.e. trade based on differences in information and not on risk considerations. Therefore, I impose risk neutrality and, as it is done in actual prediction markets, I place an exogenous limit on investment, which precludes any single agent from affecting prices. Second, I look at a market where traders post limit orders as a function of their private information but they cannot condition them on the price realization (REE models), given that agents submit their orders before observing the *actual* transaction price.[12] This restriction stems from the fact that in markets with a double auction format, a ban on short sales and tight investment limits prevent traders from (roughly) replicating a full demand schedule using an array of limit orders. In addition, the only source of uncertainty for most of the analysis is about fundamentals: sophisticated traders face uncertainty both about the asset value and about the demand of naïve traders, but the latter is a function of the asset value. Nonetheless, I provide an example in which there is also aggregate uncertainty about the fraction of naïve traders and show that, although prices are no longer fully revealing, we get the same patterns of mispricing in expectation.

The second strand of literature looks at information aggregation by modeling markets as common value auctions. Most research has focused on one-sided common value auctions, starting with the first price auction of Wilson (1977). The main finding is that equilibrium prices converge to the value of the asset as the number of

[11] Compared to noisy REE literature in this noise traders are utility maximizers endowed with sticky beliefs about future prices. Thus they play a more fundamental role than just providing liquidity.

[12] Technically, a limit order is a one-step demand/supply function whereas conditioning on the realization of the price is equivalent to submitting a full demand schedule.

bidders gets large as long as, for any two values, there exists a signal that is arbitrarily more likely at one of them (Wilson (1977) and Milgrom (1979, 1981)) or as the units at auction increase (this is the *double largeness* condition in Pesendorfer and Swinkels (1997)).[13] Information aggregation is caused by agents' inferences about future prices based on their private information and on the equilibrium behavior of the other agents. These inferences influence bidding behavior which, in turn, determines prices.

The main drawback of using one-sided auctions to analyze information aggregation in markets is that there is an implicit market maker (the seller) in charge of the supply. Double auction models solve this issue by having both strategic buyers and sellers. However, common value double auctions have proven quite intractable and very little research exists in this area. A remarkable exception is the paper by Reny and Perry (2006), who study the existence of fully revealing equilibrium prices in finite agent mixed value double auctions. They show that, when the number of agents is sufficiently large, there exits a symmetric equilibrium with prices close to the fully revealing prices of a continuum agent economy. In their model, agents' utility is strictly increasing in the signals agents privately receive. That is, traders exhibit idiosyncratic preferences that depend on the agents' private information. Since this private value component is non-negligible prices do not converge to the common value component as the market grows. Sophisticated traders in my model are pure speculators (i.e. they do not have personal tastes about the asset value) and, while I show existence and uniqueness of fully revealing equilibrium prices, the focus is on characterizing mispricing when naïve traders participate in the market.

2 The model

There is a continuum of agents. A fraction $\gamma \in (0,1)$ of them are sellers, each owning one unit of a security, with the remaining fraction being buyers, willing to buy at most one unit. The value of the security $V \in [0,1]$ is unknown with probability distribution $G(\cdot)$. Each agent receives a private signal $S \in [0,1]$ stochastically related to V.[14] Signals are independent and identically distributed conditional on v, with probability distribution $F(\cdot|v)$.

Assumption 1 *G has a C^1 density g bounded away from 0 in $[0,1]$. F is C^2 with density f bounded away from 0 in $[0,1]^2$.*

Assumption 2 *$f(\cdot|\cdot)$ satisfies the strict monotone likelihood ratio property (MLRP).*

[13] Kremer (2002) summarizes existing results and extends them to the English auction, while Hong and Shum (2004) study the rates of convergence under both scenarios.

[14] Capital letters denote random variables (V, S) and lowercase letters denote realizations (v, s).

The first assumption implies that the distribution of signals has full support for all values of the asset. That is, a trader receiving a signal $s \in [0, 1]$ cannot rule out any asset value in $[0, 1]$. The second assumption means that higher signals are more likely than lower signals when the asset value is high.

Buyers and sellers simultaneously submit bids and asks to buy and sell specifying, respectively, the maximum price willing to pay and the minimum price willing to accept. Bids are restricted to be in $[0, 1]$.[15] The price p is given by the $(1-\gamma)$-th percentile of the bid distribution. Buyers with bids above p and sellers with asks below p get to trade.[16] If there is a positive mass of bids at p there is the possibility of rationing, i.e. some traders bidding exactly p may not trade. In this case, the traders bidding p who end up with the object are chosen randomly.[17]

A fraction $\eta \in [0, 1]$ of the trader population is naïve, while the remaining population consists of risk-neutral, sophisticated traders. The bidding behavior of naïve traders is summarized by the bid distribution $H(\cdot|v)$, which is a primitive of the model.[18] Accordingly, the solution concept I use is Bayes-Nash equilibrium (BNE) in which sophisticated traders best respond to the equilibrium strategies of the other sophisticated traders, *taking the distribution of naïve bids as given.*

I assume that $H(\cdot|v)$ satisfies some regularity conditions, namely it is differentiable, weakly monotonic with respect to asset values and has the same connected support for all v.

Assumption 3 *$H(\cdot|v)$ has a density bounded above and away from zero in $[\underline{b}^H, \overline{b}^H] \subseteq [0, 1]$ for all $v \in [0, 1]$, with $\underline{b}^H < \overline{b}^H$. $H(\cdot|\cdot)$ is C^1 in $(\underline{b}^H, \overline{b}^H) \times [0, 1]$ and absolutely continuous in $[0, 1]^2$.*

This assumption implies that the distribution of naïve bids is atomless. Having a positive density implies that $H(\cdot|v)$ is strictly increasing in $(\underline{b}^H, \overline{b}^H)$ for all $v \in [0, 1]$, i.e. there are no intervals between the lowest and highest naïve bids where the mass of bids is zero.

Assumption 4 *$H(b|\cdot)$ is non-increasing in $[0, 1]$ for all $b \in [0, 1]$.*

Assumption 4 implies that, for all v, v' such that $v > v'$, $H(\cdot|v)$ first order stochastically dominates $H(\cdot|v')$. It means that naïve traders tend (weakly) to bid higher when the value of the asset is higher.

[15] This assumption is without loss of generality since bids outside the unit interval are weakly dominated by bidding either zero or one.

[16] In the remainder of the paper I use the term "bid" to refer both to seller asks and buyer bids.

[17] Reny and Perry (2006) use the same tie-breaking rule.

[18] Notice that in this continuum economy, given the price mechanism of the double auction, prices depend on the fraction of naïve traders but not on how they are distributed across buyers and sellers. Thus, I do not make any assumptions on the proportion of naïve traders that are sellers. Also, I do not require the distributions of naïve buyers' and sellers' bids to be identical.

Modeling the distribution of naïve bids rather than imposing conditions on the bidding strategies of naïve traders provides a high level of generality to the results shown below. This is because most models of boundedly rational traders proposed in finance models as well as in behavioral game theory will result in bid distributions satisfying *Assumptions 3* and *4*. Examples include noise or liquidity traders, who bid randomly ($H(b|v) = b$ for all v), and traders bidding their interim valuations, $\mathbb{E}(V|s)$. In a continuum agent economy, the latter represent *fully cursed* traders, who fail to account for the common value nature of the asset (Holt and Sherman (1994), Kagel and Levin (1986) and Eyster and Rabin (2005)), and also agents who mistakenly believe that everybody shares their own information structure (in Jehiel and Koessler (2008) definition of analogy-based expectation equilibrium, these agents correspond to the *private information* analogy partition).[19] In addition, the above assumptions can accommodate a mix of level-k agents, which represents a population with different degrees of bounded rationality, i.e. a cognitive hierarchy (see Crawford and Iriberri (2007) for a definition of level-k thinking in auctions).[20] The critical restriction I impose is that, H being a primitive of the model, naïve bidding is not the result of (naïvely) responding to sophisticated traders' *equilibrium* strategies.[21]

Let $\mathcal{T}$ denote the subset of sophisticated traders. Given a profile of bidding (pure) strategies $\beta : [0, 1] \times \mathcal{T} \to [0, 1]$ with $\beta(s, t)$ denoting the bid of sophisticated trader $t \in \mathcal{T}$ when she receives signal s, let $B(p|V, \eta)$ be the mass of bids lower than or equal to p when the share of naïve traders is η and $B\_(p|V, \eta)$ the mass of bids strictly less than p. Accordingly,

$$B(p|v, \eta) := \eta H(p|v) + (1 - \eta) \int_{\mathcal{T}} \int_0^1 1_{\{\beta(s,t) \leq p\}} f(s|v) ds d\mu, \tag{1}$$

and

$$B\_(p|v, \eta) := \eta H(p|v) + (1 - \eta) \int_{\mathcal{T}} \int_0^1 1_{\{\beta(s,t) < p\}} f(s|v) ds d\mu, \tag{2}$$

where μ is a suitable (atomless) measure on $\mathcal{T}$ and $1_{\{\cdot\}}$ is the indicator function.

[19] This behavior is also similar to the one exhibited by traders in the prediction market models of Manski (2006), Gjerstad (2005) and Wolfers and Zitzewitz (2006).

[20] More precisely, one can find a bid distribution satisfying assumptions 3 and 4 that is arbitrarily close to the bid distribution of a population level-k agents. This is so because the distribution of bids generated by such population can include atoms and therefore violate *Assumption 3*.

[21] This does not mean that naïve traders need not best respond to some strategy. For instance, a level-k trader best responds to the bidding strategy of a level-$(k-1)$, implying that H is determined by the bidding rule of level-0 agents and the proportions of each level in the naïve population. In other words, equilibrium naïve strategies are not obtained through fixed point arguments as it is the case with sophisticated traders' strategies. This approach rules out some behavioral types such as partially cursed traders (Eyster and Rabin (2005)) or the naïve traders in Esponda (2008), whose equilibrium strategies represent a fixed point of a boundedly rational best responses.

Given that there is a continuum of traders receiving i.i.d. signals, by the law of large numbers, the profile of signals received by traders coincides with the whole distribution of signals conditional on V, $F(\cdot|V)$.[22] Accordingly, given strategy profile $\beta(\cdot,\cdot)$, the market clearing price is completely determined by the realization of V (and the share η). Hence, for all $v \in [0,1]$ the market price is given by the function $\rho : [0,1]^2 \to [0,1]$ that satisfies

$$(1-\gamma) \in [B\_(\rho(v,\eta)|v,\eta), B(\rho(v,\eta)|v,\eta)]. \tag{3}$$

The payoff functions for a sophisticated buyer t and a seller t' are, respectively,

$$\begin{aligned} \pi^{buy}(s,t) \ &:= \ \mathbb{E}((V-\rho(V,\eta))1_{\{\beta(s,t)>\rho(V,\eta)\}}|s) \\ &+ \ \mathbb{E}((V-\beta(s,t))\lambda(\beta(s,t),V)1_{\{\beta(s,t)=\rho(V,\eta)\}}|s), \end{aligned} \tag{4}$$

and

$$\begin{aligned} \pi^{sell}(s,t') \ &:= \ \mathbb{E}((\rho(V,\eta)-V)1_{\{\beta(s,t')<\rho(V,\eta)\}}|s) \\ &+ \ \mathbb{E}((\beta(s,t')-V)(1-\lambda(\beta(s,t'),V))1_{\{\beta(s,t')=\rho(V,\eta)\}}|s), \end{aligned} \tag{5}$$

where $\lambda(b,v)$ represents the probability of getting the object given bid b and asset value v when $\rho(v,\eta) = b$.

3 Equilibrium Prices

In this section I investigate how well equilibrium prices match asset values as a function of the fraction of naïve traders (η) and of their bidding behavior (H). Accordingly, I restrict my attention to equilibria with prices $\rho(v,\eta)$ that are increasing in v (henceforth monotone equilibria). The two main results are stated in *Propositions 1* and *2*. The first provides a characterization of monotone equilibrium prices, whereas the second shows existence and uniqueness of such prices. All proofs are relegated to the *Appendix*.

The characterization of equilibrium prices provided below is driven by the inability of a single trader to affect prices when there is a continuum of agents. Price taking behavior induces two key features of payoff functions (4)-(5): (i) buyers and sellers have the same preference ranking over bids (*Lemma 1*), and (ii) bidding behavior is oriented to maximize the probability of trading in favorable conditions

[22]As pointed out by Judd (1985) there are measurability problems when dealing with a continuum of random variables. While acknowledging those issues, I do not address them in the analysis presented here. Hammond and Sun (2006) propose extending the usual product probability space to one that retains the Fubini property so that measurability is restored. An alternative approach is to have countably many agents with charge spaces (Feldman and Gilles (1985)).

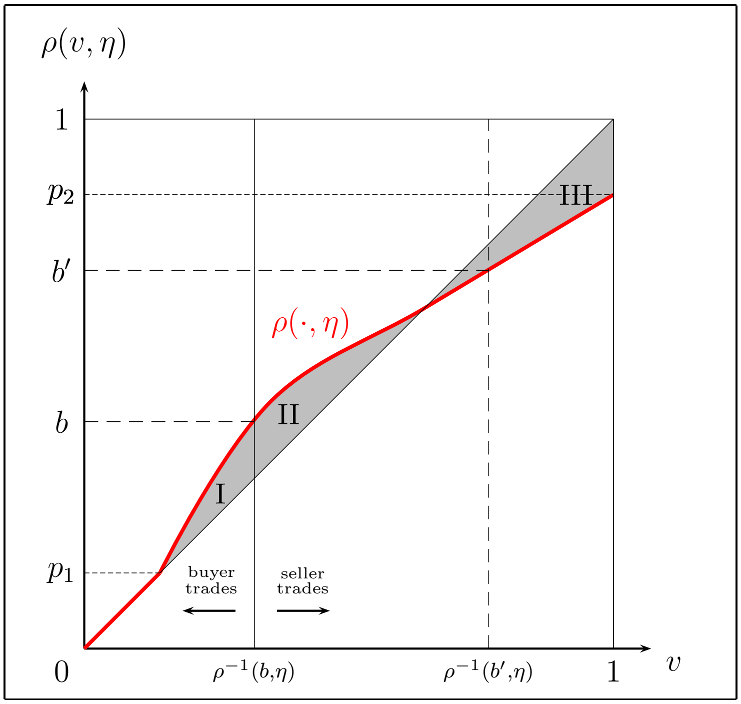

Figure 1: Sophisticated Bidding

while avoiding undesired trades, considering prices fixed. The latter, coupled with increasing prices, leads sophisticated traders to avoid bidding in areas of the price range where prices are not equal to asset values, i.e. where there is mispricing.

Lemma 1 (Symmetric Preferences) *Buyers and sellers receiving the same signal $s \in [0,1]$ have the same preferences over bids.*

To get some intuition on both the symmetry of preferences and the incentive to avoid bidding in mispricing regions, consider the price function depicted in *Figure 1*. The range of prices consists of a interval with no mispricing $[0, p_1]$ and a mispricing interval $[p_1, p_2]$. Prices are greater than values whenever $\rho(v, \eta)$ is above the diagonal and viceversa. Assuming there is no rationing, the payoff for a buyer bidding b when she receives signal s is the expected value of the difference $v - \rho(v, \eta)$, conditional on s and on prices being below b. That is, it is the negative of the expected value of shaded area I. A seller with signal s and bid b, gets the expected value of $\rho(v, \eta) - v$ for prices above b, i.e. the expected difference between transactions involving prices above values (area II) and transactions for which $\rho(v, \eta) < v$ (area III).

The rationale behind symmetric preferences is the following: for a seller with signal s, the payoff for trading the object when prices fall within two alternative bids b and b' is the negative of the payoff for a buyer receiving the same signal. In addition, when a seller bidding b trades (because $\rho(v, \eta) > b$) a buyer bidding b does not trade. If a seller with signal s strictly prefers bid b to bid $b' > b$ it is because the expected payoff, conditional on s, of trading at prices between b and b' is positive.[23] But then, a buyer with the same signal would rather avoid trading at those prices by also bidding b.

Consider now a seller who places a bid in a mispricing area, e.g. by bidding b. She has an incentive to deviate and bid either in $[0, p_1]$ since all transactions in area I involve $\rho(v, \eta) > v$, or to bid in $[p_2, 1]$ if, conditional on her signal, the expected difference between the gains in areas I and II and the losses in area III is negative. In the latter case, by bidding above p_2 she abstains from trading and gets zero payoffs. On the other hand, if a seller bids b' she is engaging in negative transactions, which can be avoided by bidding above the area where $\rho(v, \eta) < v$ (i.e. in $[p_2, 1]$) or can be compensated with gains from areas I and II (by bidding in $[0, p_1]$). By the symmetry of preferences, no buyer would bid b or b'. This also indicates that no sophisticated trader would bid below a mispricing interval that starts with $\rho(v, \eta) < v$.

Proposition 1 and *Corollary 1* are a direct consequence of these two key characteristics of traders' payoffs. Let $H(v) = H(v|v)$.

Proposition 1 (Equilibrium Prices) *In any monotone equilibrium of a CVDA satisfying Assumptions 1-4, there is a set* $\mathcal{V} = \bigcup\limits_k [\underline{v}_k, \overline{v}_k]$ *with* $\overline{v}_k < \underline{v}_{k+1}$ *for all* $k = 1, ..., K \leq \infty$ *and a collection of signals* $\{s_k^*\}$ *with* $s_k^* < s_{k+1}^*$ *such that prices are given by*

$$\rho(v, \eta) = \begin{cases} v & \text{if } v \in [0, 1] \setminus \mathcal{V} \\ p \text{ s.t. } H(p|v) = \frac{1-\gamma-(1-\eta)F(s_k^*|v)}{\eta} & \text{if } v \in [\underline{v}_k, \overline{v}_k], \end{cases} \tag{6}$$

where all $\underline{v}_k, \overline{v}_k \in (0, 1)$ *and* s_k^* *satisfy:*

$$1 - \gamma = \eta H(\underline{v}_k) + (1 - \eta) F(s_k^*|\underline{v}_k), \tag{7}$$

$$1 - \gamma = \eta H(\overline{v}_k) + (1 - \eta) F(s_k^*|\overline{v}_k), \tag{8}$$

and, for all $s \leq s_k^*$ $(s \geq s_k^*)$,

$$\mathbb{E}((V - \rho(V, \eta)) 1_{\{V \in (\underline{v}_k, \overline{v}_k)\}} | s) \leq 0 \ \ (\geq 0). \tag{9}$$

This result essentially describes monotone equilibria as the succession of mispricing intervals ($[\underline{v}_k, \overline{v}_k]$), where prices are determined by the naïve bids, and intervals with no mispricing ($[\overline{v}_k, \underline{v}_{k+1}]$) in which all sophisticated bids within the price range are concentrated. To see how, notice that $F(s_k^*|v)$ represents in (6) the mass of sophisticated bids below $v \in [\underline{v}_k, \overline{v}_k]$. Since s_k^* does not change with v, sophisticated traders with signals below s_k^* bid below $\underline{v}_k$ and traders with signals above s_k^* bid

[23] That is, for values between $\rho^{-1}(b, \eta)$ and $\rho^{-1}(b', \eta)$, where ρ^{-1} denotes the inverse image.

above $\overline{v}_k$. It also establishes that the allocation of sophisticated bids across the latter intervals is block-monotonic, i.e. lower signal traders bid in lower intervals. Specifically, traders with signals in (s^*_k, s^*_{k+1}) bid in $[\overline{v}_k, \underline{v}_{k+1}]$.[24] Block-monotonicity means that the sophisticated traders more active in the market are the low signal sellers and the high signal buyers. Since the former tend to bid relatively low and the latter bid relatively high, they engage in trade more often, compared to high signal sellers and low signal buyers.

It is important to point out that there may be no mispricing at all, i.e. $\mathcal{V}$ is the empty set, or mispricing almost everywhere, with prices completely determined by the distribution of naïve bids, i.e. $\mathcal{V} = [0,1]$. The next corollary further requires that, in any mispricing interval, prices are above values in the lower portion of the interval and below values in the upper part. This over-then-underpricing can be seen as a local favorite-longshot bias, since it is confined in mispricing intervals.

Corollary 1 *In any monotone equilibrium with $\mathcal{V}$ non-empty, given $(\underline{v}_k, \overline{v}_k)$, either $\rho(v,\eta) = v$ a.e. in $(\underline{v}_k, \overline{v}_k)$ or there exist v'_k, v''_k with $\underline{v}_k < v'_k \leq v''_k < \overline{v}_k$ such that $\rho(v,\eta) \geq v$ a.e. in $[\underline{v}_k, v'_k]$ with strict inequality in a non-null subset, and $\rho(v,\eta) \leq v$ a.e. in $[v''_k, \overline{v}_k]$ with strict inequality in a non-null subset. Moreover, if this is true for intervals $(\underline{v}_k, \hat{v}]$ and $(\hat{v}, \overline{v}_k)$, then $\mathbb{E}((V - \rho(V,\eta))1_{\{V \in (\hat{v}, \overline{v}_k)\}} | s^*_k) \geq 0$ if $s^*_k < 1$.*

Corollary 1 states that prices in intervals where no sophisticated bids are placed need to begin with prices above values and end with prices below values. When faced with the prospect of bidding just below an interval where values are always above prices, a seller would rather deviate and bid just above that region to avoid trading at those prices. A symmetric reasoning applies when the seller bids just above an interval in which $\rho(v,\eta) > v$. It also says that, if a mispricing interval consists of two or more disjoint subintervals, each of them beginning with prices above values and ending with prices below values, no sophisticated bidder bidding below such interval has an incentive to deviate and bid in between two of those subintervals.

Figure 2 shows prices that can and cannot be equilibrium prices.

The proofs of *Proposition 1* and *Corollary 1* hinge upon a series of lemmas in the *Appendix*, which formalize the intuition about sophisticated bidding presented above. In addition to having symmetric preferences (*Lemma 1*), I show that no sophisticated bidder would place bids just below a mispricing interval that starts with prices below values (*Lemma 2*). *Lemma 3* states that sophisticated bidders avoid placing bids in mispricing intervals. Finally, as assumed above, no rationing of sophisticated traders takes place in equilibrium, since any atom is solely created by naïve traders and can only happen in very special cases (*Lemma 4*). In addition, the block-monotonicity of the distribution of sophisticated bids is a direct consequence

[24] When $\underline{v}_1 > 0$ traders with signals below s^*_1 bid in $[0, \underline{v}_1]$ and in $[0, \rho^{-1}(0,\eta)]$ if $\underline{v}_1 = 0$, i.e. outside the price range. Similarly, traders with signals above s^*_K either bid in $[\overline{v}_K, 1]$ (when $\overline{v}_K < 1$) or in $[\rho^{-1}(1,\eta), 1]$ (when $\overline{v}_K = 1$).

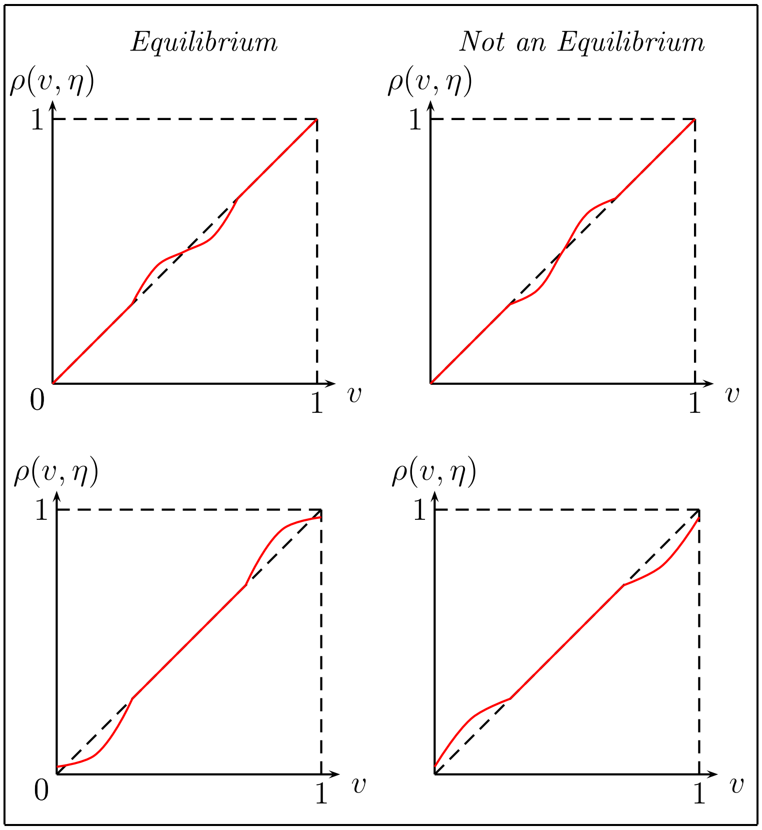

Figure 2: Candidates for Equilibrium Prices

of the MLRP property and of *Lemma 2*: if bidding just below a mispricing interval is profitable for a trader with signal s, it is also profitable for all traders with signals below s.

The next result states that monotone equilibria exist in any CVDA with a continuum of agents satisfying *Assumptions 1-4*. Furthermore, monotone equilibrium prices are essentially unique. Finally, it sheds light on how the presence of naïve traders affect the informational content of prices: there is a strictly positive lower bound on the share of naïve bidders below which there is no mispricing and there is an upper bound above which prices are always set by naïve bidders while sophisticated bidders bid outside the price range.

Proposition 2 (Existence of Monotone Equilibria) *Let Assumptions 1-4 be satisfied. Then a monotone Bayesian Nash equilibrium in pure strategies exists for all* $\eta \in [0,1]$ *and the resulting price function* $\rho(\cdot,\eta)$ *is essentially the unique monotone price function that can be supported in equilibrium. Furthermore, there exists* $\underline{\eta} \in (0, \min\{\gamma, 1-\gamma\})$ *such that* $\mathcal{V}$ *is the empty set for all* $\eta \leq \underline{\eta}$*, and there exists* $\overline{\eta} \leq 1$ *such that* $\mathcal{V} = [0,1]$ *for all* $\eta > \overline{\eta}$ *(*$\overline{\eta} < 1$ *if* $H'(v) \geq 0$ *whenever* $H(v) = 1-\gamma$*).*

Existence of equilibrium is given by the continuity of distributions H, F and expectations, and by the monotonicity of H, F with respect to v. The former guarantees the existence, for each η, of a block-monotonic distribution of sophisticated bids satisfying the equilibrium conditions of *Proposition 1.* The latter leads to increasing prices when the distribution of sophisticated bids is block-monotonic. Uniqueness is based on the fact that, due to the strict MLRP, each triplet $(s_k^*, \underline{v}_k, \overline{v}_k)$ satisfying (7)-(9) and *Corollary 1* is unique. An algorithm to obtain the collection $\{(s_k^*, \underline{v}_k, \overline{v}_k)\}_{k=1}^K$ that characterizes equilibrium price $\rho(v, \eta)$ is provided in the proof of *Proposition 2.*

The last part of *Proposition 2* establishes the existence of three types of equilibrium prices depending on the proportion of naïve traders: no mispricing, partial mispricing (i.e. with both mispricing and correct pricing intervals) and complete mispricing. To provide some intuition on this result, let the quantile function $\alpha(v, \eta)$ represent the highest signal corresponding to bids at or below v such that $\rho(v, \eta) = v$, assuming block-monotonicity. That is, for all $v \in [0, 1]$ such that $\frac{1-\gamma-\eta H(v)}{1-\eta} \in [0, 1]$, $\alpha(v, \eta)$ is given by

$$F(\alpha(v, \eta)|v) = \frac{1 - \gamma - \eta H(v)}{1 - \eta}. \tag{10}$$

For most distributions of naïve bids, $\alpha(\cdot, \cdot)$ has three distinct regions, depending on the value of η (see *Lemma 5* in the *Appendix*).[25] If $\eta \in [0, \underline{\eta}]$, it is increasing with respect to v in $[0, 1]$. It is non-monotonic (whenever it is well-defined) in v for $\eta \in (\underline{\eta}, \overline{\eta})$. Finally, it is decreasing for all $\eta \geq \underline{\eta}$. I show that, when $\alpha(\cdot, \eta)$ is increasing everywhere, prices must equal values. If there were a mispricing interval $[\underline{v}_k, \overline{v}_k]$, then $s_k^* > \alpha(\underline{v}_k, \eta)$ and/or $s_k^* < \alpha(\overline{v}_k, \overline{v}_k)$ given that $\alpha(\cdot, \eta)$ is increasing. But this means that prices are below values in the lower part of $[\underline{v}_k, \overline{v}_k]$ and/or above values in the upper part of the interval, thus violating *Corollary 1.* On the other hand, there must be mispricing in intervals of values where $\alpha(\cdot, \eta)$ is not well-defined or decreasing (*Lemma 6*). For prices to be equal values in some interval $[v_1, v_2]$ in which $\alpha(\cdot, \eta)$ is decreasing, it would be necessary to have a mass of bids below v_2 that is strictly smaller than the mass of bids below $v_1 < v_2$. Moreover, that reduction of mass needs to be greater than $F(\alpha(v_1, \eta)|v_1) - F(\alpha(v_1, \eta)|v_2)$, given that $\alpha(v_2, \eta) < \alpha(v_1, \eta)$. However, this is not possible under the strict MLRP, because the highest possible reduction of mass is obtained by having sophisticated bid distributions equal to $F(\alpha(\cdot, \eta)|\cdot)$.

It is important to point out that, although $\rho(\cdot, \eta)$ is essentially unique, there are in general multiple equilibria associated with $\rho(\cdot, \eta)$. That is, multiple profiles of sophisticated bidding strategies supporting $\rho(\cdot, \eta)$, including asymmetric equilibria and even equilibria in which a positive mass of traders follow non-monotonic bidding strategies.[26] The following bidding strategy corresponds to the symmetric

[25] Distributions $H(\cdot|\cdot)$ such that $H'(v) < 0$ for some v are typically multi-modal distributions, with most of the mass concentrated in a small subset of the support.

[26] An example of such equilibrium can be provided by the author upon request. The important

equilibrium given by[27]

$$\beta(s,t) = \begin{cases} 0 & \text{if } s < \min\{\alpha(0,\eta), s_1^*\} \\ v \in [0, \underline{v}_1) \text{ s.t. } \alpha(v,\eta) = s & \text{if } s \in [\alpha(0,\eta), s_1^*) \\ v \in [\overline{v}_k, \underline{v}_{k+1}] \text{ s.t. } \alpha(v,\eta) = s & \text{if } s \in [s_k^*, s_{k+1}^*] \\ v \in (\overline{v}_K, 1] \text{ s.t. } \alpha(v,\eta) = s & \text{if } s \in (s_K^*, \alpha(1,\eta)] \\ 1 & \text{if } s > \max\{s_K^*, \alpha(1,\eta)\} \end{cases} \tag{11}$$

I now provide an example to illustrate how the extent of mispricing varies with the share of naïve traders and to get some intuition for *Proposition 2*.

Example 1 *Consider a CVDA with the following characteristics. V is distributed uniformly in $[0,1]$; the conditional distribution of signals is Beta$(1+v,1)$ (i.e. $F(s|v) = s^{1+v}$);*[28] *each naïve trader bids according to $\beta^n(s) := \frac{3}{5}s^{1/5}$, which is a rough approximation of bidding $\mathbb{E}(V|s)$.*[29]

Given $\beta^n(\cdot)$, the distribution of naïve bids is

$$H(p|v) = \begin{cases} \left(\frac{5}{3}p\right)^{5(1+v)} & \text{if } v \le \frac{3}{5} \\ 1 & \text{if } v > \frac{3}{5} \end{cases} \tag{12}$$

By *Proposition 2*, there exist cutoff points $\underline{\eta}$, $\overline{\eta}$ that determine whether there will be no, partial or complete mispricing as a function of η. Since $H'(v) \ge 0$ for all v, $\overline{\eta}$ is strictly less than one.

The first thing to note is that, given η, a necessary condition for partial mispricing with $\mathcal{V} = [\underline{v}_1, \overline{v}_1]$ is that there exist a signal s_1^* satisfying (7) at three distinct values, namely $\underline{v}_1$, $\overline{v}_1$ and $v_1' \in (\underline{v}_1, \overline{v}_1)$, the latter being the point at which $\rho(v,\eta)$ goes from being above to go below v. Therefore, the function $\alpha(v,\eta)$ given by

$$\alpha(v,\eta) = F^{-1}(\tfrac{1-\gamma-\eta H(v)}{1-\eta}|v) = \left[\frac{1-\gamma-\eta(1_{\{v>3/5\}} + 1_{\{v\le 3/5\}}\left(\frac{5}{3}v\right)^{5(1+v)})}{1-\eta}\right]^{\frac{1}{1+v}}$$

needs to be three-to-one in some subset of its range. If it is strictly increasing in $[0,1]$, then equilibrium prices will necessarily equal to values. On the other hand, if

restriction is that bidding strategies be block-monotonic and that the resulting bid distribution is block monotonic. Thus, some traders may follow strictly decreasing bid functions inside intervals with no mispricing.

[27] If $\mathcal{V}$ is the empty set, let $s_1^* = \alpha(1,\eta)$ and $\underline{v}_1 = 1$.

[28] This distribution satisfies all assumptions except the full support, provided it has positive density in $(0,1)$ rather than in $[0,1]$.

[29] This approximation makes computations more tractable without changing any substantive aspect of the analysis.

for some η there exists a signal s such that $\mathbb{E}(v-\rho(v,\eta)|s)=0$ where $\rho(v,\eta)$ is given by $1-\gamma=\eta H(\rho(v,\eta)|v)+(1-\eta)F(s|v)$ for all $v\in[0,1)$ and satisfies $\rho(0,\eta)>0$ and $\rho(1,\eta)<1$, then $[v_1,v_2]=[0,1]$ fulfils *Corollary 1*, and sophisticated bids will be confined to $[0,1]\setminus(\rho(0,\eta),\rho(1,\eta))$.[30]

In a symmetric market ($\gamma=0.5$), I find that $\underline{\eta}\approx 0.016$ and $\overline{\eta}\approx 0.214$. This shows that the range of η compatible with perfect prices can be quite small. As an illustration, the top graph in *Figure 3's* left panel shows equilibrium prices when 10% of traders are naïve. Even with such a low proportion of naïve traders, the probability that prices reflect the true asset value is roughly one half in this example.

The graph of $\alpha(v,0.1)$ (middle graph in the left panel of *Figure 3*) provides some intuition on the existence and uniqueness of prices. As mentioned above, $(s_1^*,\underline{v}_1,\overline{v}_1)$ are given by (7)-(9), that is $\rho(\underline{v}_1,0.1)=\underline{v}_1$, $\rho(\overline{v}_1,0.1)=\overline{v}_1$ and $\mathbb{E}((V-\rho(V,\eta)1_{\{v\in[\underline{v}_1,\overline{v}_1]\}}|s_1^*)=0$. The latter implies that the expected gain a seller with signal s_1^* makes when she trades at $\rho(v,\eta)>v$ is exactly offset by trades at $\rho(v,\eta)<v$: these two regions are given by $[\underline{v}_1,v_1')$ and $(v_1',\overline{v}_1]$, respectively. Looking at the graph of $\alpha(v,0.1)$ we can see that, as s_1^* increases, the distance between $\underline{v}_1$ and v_1' goes to zero implying that the set of trades with positive payoff shrinks to zero. Similarly, the distance between v_1' and $\overline{v}_1$ goes to zero when s_1^* decreases. Therefore, by the continuity of $\mathbb{E}(\cdot|\cdot)$ and $\alpha(\cdot,0.1)$, we can find a unique triplet $(s_1^*,\underline{v}_1,\overline{v}_1)$ satisfying the conditions of *Proposition 1* and *Corollary 1*.

To complete the example, the bottom graph of *Figure 3's* left panel shows symmetric equilibrium bidding strategies implementing $\rho(\cdot,0.1)$.

3.1 Prices when the fraction of naïve traders is unknown

An important characteristic of the model is that, unlike existing models of noisy REE, there is no aggregate uncertainty about the demand of naïve traders besides v.[31] This implies that, except in special cases, prices are fully revealing. It would be interesting to know what happens to prices and sophisticated traders' bidding strategies when there is an additional source of aggregate uncertainty. A natural way to introduce such uncertainty is to make the fraction of naïve traders a random variable whose distribution is known by sophisticated traders.[32] As the next example shows, although prices are no longer fully revealing, sophisticated traders' behavior is remarkably similar to the case of η being common knowledge and the results stated

[30] If $\mathbb{E}(V-\rho(V,\eta)|0)\geq 0$ all the mass of risk-neutral bids would be placed above $\rho(1,\eta)$ whereas it would be placed below $\rho(0,\eta)$ when $\mathbb{E}(V-\rho(V,\eta)|1)\leq 0$.

[31] For instance, in Hellwig (1980) and Kyle (1989), as the number of agents increases, the variance of the distribution of noise trade demand goes to infinity. Since sophisticated traders are strictly risk averse their demands are finite, implying that the uncertainty about the aggregate demand of noise traders does not vanish in the limit and prices are not fully revealing.

[32] That is, there are shocks to the aggregate demand of naïve traders and, in turn, to that of sophisticated traders.

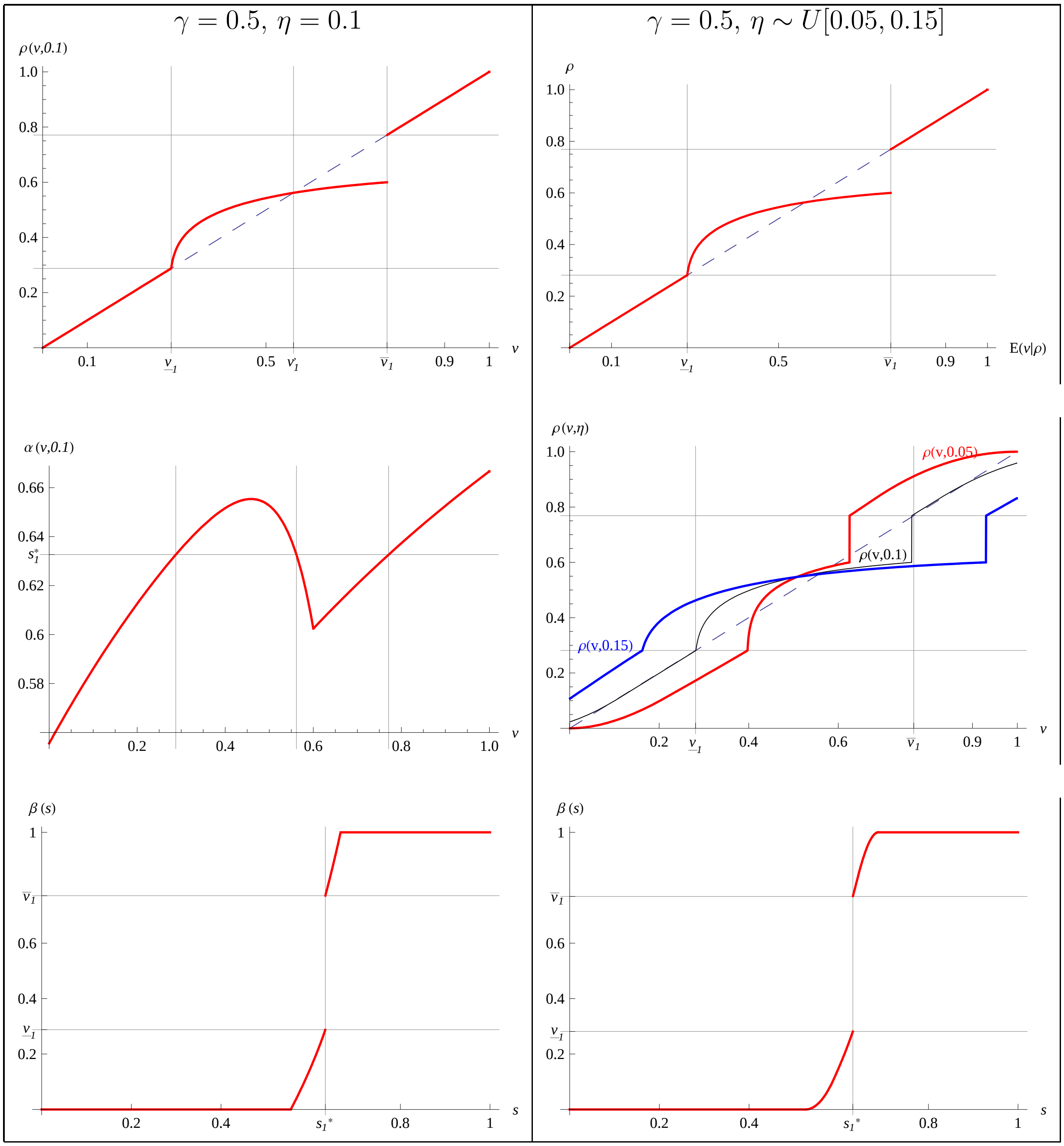

Figure 3: Equilibrium prices and bidding strategies.

above continue to hold *in expectation*. Specifically, there are intervals in the price range where sophisticated traders avoid placing their bids, which are associated with expected mispricing, i.e. $\mathbb{E}(V|\rho(V,\widetilde{\eta}) = p) \neq p$ ($\widetilde{\eta}$ denotes the random fraction of naïve traders). In any such interval there is (expected) overpricing ($\mathbb{E}(V|\rho(V,\widetilde{\eta}) = p) < p$) in the lower part of the interval and underpricing in the upper part.[33] This result follows from the same type of argument used to prove *Proposition 1* and *Corollary 1*: if the equilibrium bid $\beta(s)$ of a sophisticated trader is inside the range of prices then it has to satisfy $\beta(s) = \mathbb{E}(V|\rho(V,\widetilde{\eta}) = \beta(s))$, which is the equivalent of the requirement that $v = \rho(v,\eta)$ whenever $\beta(s) = \rho(v,\eta)$ in the case of known η.

In addition, it can be shown that there exist probability distributions over $\widetilde{\eta}$ with support bounded away from zero for which there is no expected mispricing in equilibrium. Unlike the competitive limits of Hellwig (1980) and Kyle (1989),[34] where there is not expected mispricing for any finite level of *per capita* noise, in this model it is possible to have no (expected) mispricing at all, mispricing for almost all prices and a combination of price intervals with and without mispricing. These differences are caused by the restrictions the trade institution places on the demand functions of sophisticated traders. In the double auction analyzed here traders can only demand/supply one unit and their demands are step functions rather than full demand/supply schedules, which limits their ability to arbitrage away any expected mispricing.

Example 2 *Consider the CVDA of Example 1 with $\widetilde{\eta}$ distributed uniformly in $[0.05, 0.15]$ and independent of V.*

Let $\hat{\alpha}(p)$ be the quantile function such that $\mathbb{E}(V|\rho(V,\widetilde{\eta}) = p) = p$ for all $p \in [0,1]$ with $\rho(V,\widetilde{\eta})$ being the market price when all sophisticated traders with signals below (above) $\hat{\alpha}(p)$ bid below (above) $\mathbb{E}(V|\rho(V,\widetilde{\eta} = p)$. In this example, $\hat{\alpha}(p)$ is non-monotonic, implying that in equilibrium there will be expected mispricing in some interval. This interval is given by the solution to the following system of equations, which resembles the conditions in *Proposition 1*:

$$
\begin{aligned}
&\mathbb{E}((V - \rho(V,\widetilde{\eta}))1_{\{\rho(V,\widetilde{\eta})\in[\underline{v}_1,\overline{v}_1]\}}|s_1^*) = 0 \\
&\mathbb{E}(V|\rho(V,\widetilde{\eta}) = \underline{v}_1) = \underline{v}_1 \\
&\mathbb{E}(V|\rho(V,\widetilde{\eta}) = \underline{v}_2) = \underline{v}_2,
\end{aligned}
$$

where $\rho(V,\widetilde{\eta})$ satisfies $1-\gamma = \widetilde{\eta}H(\rho(V,\widetilde{\eta})|V) + (1-\widetilde{\eta})F(s_1^*|V)$. The top graph in the right panel of *Figure 3* shows the mapping from conditional expected values to prices, which is remarkably similar to prices when the fraction of naïve traders is known and equal to 0.1. The middle graph shows prices for various realizations of $\widetilde{\eta}$. Equilibrium

[33]Other features of the analysis continue to hold, namely that atoms in the distribution of prices cannot be caused by sophisticated traders for the same reasons as in the case where η is known.

[34]See *Theorem 9.2* in Kyle (1989).

prices will lie in the area between $\rho(\cdot, 0.05)$ and $\rho(\cdot, 0.15)$. Interestingly, for low realizations of V, prices will be higher than values and viceversa (i.e. $\mathbb{E}(\rho(V, \widetilde{\eta})|v) > (<)\ v$ when v is close to zero (one)), even though there is no expected mispricing at low and high prices.[35]

Finally, the bottom graph depicts the discontinuous bidding strategy used by sophisticated traders in a symmetric equilibrium, which is given by

$$\beta(s) = \begin{cases} 0 & \text{if } s < \frac{0.5}{0.95} \\ p \in [0, \underline{v}_1) \text{ s.t. } \hat{\alpha}(p) = s & \text{if } s \in \left[\frac{0.5}{0.95}, s_1^*\right) \\ p \in (\overline{v}_1, 1] \text{ s.t. } \hat{\alpha}(p) = s & \text{if } s \in \left[s_1^*, \sqrt{\frac{0.45}{0.95}}\right) \\ 1 & \text{if } s \geq \sqrt{\frac{0.45}{0.95}}. \end{cases} \tag{13}$$

4 Empirical identification of mispricing

In most asset markets fundamental values are not observed. This poses a challenge to the empirical identification of mispricing in those markets. Assuming the model laid out in the previous sections is a reasonable approximation of some existing asset markets,[36] it is possible to identify empirically the intervals of prices where $\rho(v, \eta) = v$ (or $\mathbb{E}(V|\rho(V, \widetilde{\eta}) = p) = p$ if there is aggregate uncertainty about η) without the need to observe v. Furthermore, this identification should not require stringent assumptions on the parameters of the market, namely $\gamma, \eta, H(\cdot|\cdot), F(\cdot|)$ and $G(\cdot)$. The identification restrictions (other than *Assumptions 1-4*) are the monotonicity of equilibrium prices and the distribution of naïve bids having full support on the set of possible asset values.[37]

The heuristics of how to identify mispricing intervals are as follows. Recall that sophisticated bids are placed only in intervals of the price range where $\rho(v, \eta) = v$. Accordingly, the density of bids is higher in a small neighborhood of the observed price when it equals the unknown asset value than when value and price differ. That is, we should observe a discontinuous change in the density of bids at the boundaries $\{\underline{v}_k\}$ and $\{\overline{v}_k\}$ of mispricing intervals. Specifically, as v increases, the density drops at $\{\underline{v}_k\}$ and jumps at $\{\overline{v}_k\}$, respectively. Moreover, as the above example

[35] Actually, it is the requirement that $\mathbb{E}(V|\rho(V, \widetilde{\eta}) = p) \leq p$ in equilibrium what causes it, given that if $\mathbb{E}(\rho(0, \widetilde{\eta})|0) = 0$ then $\rho(0, \eta) = 0$ for almost all η in the support and, by continuity of H and F, there exists a non-degenerate interval of values for which prices are zero with positive probability, implying that $\mathbb{E}(V|\rho(V, \widetilde{\eta}) = 0) > 0$.

[36] Apart from prediction markets, other trading institutions with a double-auction format such as call markets, futures markets or stock exchanges could be suitable for this empirical approach as long as participants in those markets can reasonably characterized as either naïve traders and arbitrageurs.

[37] Obviously, since this is an static model and most actual markets are dynamic, any empirical analysis would require strong stationarity assumptions.

shows, these discontinuities exist even when sophisticated traders do not observe η. Hence, using a series on prices and depth of the order book one could statistically distinguish the two pricing regimes by estimating the discontinuity points in the density of bids. Several kernel methods have been proposed to estimate locations of discontinuities in density functions and also in a regression context (see Chu and Cheng (1996), Delaigle and Gijbels (2006) and references therein).[38] A major caveat is that the fraction of naive traders needs to be such that both mispricing and accurate pricing intervals are present. This is so because discontinuities (other than at the boundaries) do not arise if there is mispricing for almost all v or if $\rho(v, \eta) = v$ for all v. Additional information would be required to distinguish between the latter two cases.

Note also that by identifying the sets where prices differ from values we can establish a (not necessarily tight) upper bound on $|\rho(v, \eta) - v|$. Assume $[\rho(\underline{v}_k, \eta), \rho(\overline{v}_k, \eta)]$ is a mispricing interval. Then, $\rho(\underline{v}_k, \eta) = \underline{v}_k$ and $\rho(\overline{v}_k, \eta) = \overline{v}_k$. By monotonicity of prices, $|\rho(v, \eta) - v| < \rho(\overline{v}_k, \eta) - \rho(\underline{v}_k, \eta)$ for all $p \in [\rho(\underline{v}_k, \eta), \rho(\overline{v}_k, \eta)]$.

5 Discussion and Concluding Remarks

In the large common value market presented here information aggregation is driven by traders' use of private information to forecast prices, even though no individual signal adds new information to the price. The degree of mispricing depends crucially on a substantial presence of sophisticated traders in the market, given that they set prices equal to values in the range of prices where they place their bids. Whenever there are not enough sophisticated traders to arbitrage away any potential mispricing, prices exhibit a *local* favorite-longshot bias: within each mispricing interval, the asset is overpriced for low values and underpriced for high values.

These results are based on several restrictive assumptions. Among others, I impose risk-neutrality, unit demand/supply, exogenous buyer/seller roles and one bid/ask per trader. I briefly discuss how they can be relaxed.

Risk aversion: the key feature of sophisticated bidding behavior, namely that they do not bid in mispricing areas of the equilibrium price function, still holds regardless of agents' attitudes toward risk since it is just a consequence of maximizing expected gains from trade when trading at prices different from values. Accordingly, one should be able to prove results similar to those presented above for a population of risk averse (or risk loving) agents.[39]

[38] The depth of the order book should be adjusted using trade volume or similar measures to account for changes in the density of bids due to changes in market size.

[39] Risk aversion breaks the symmetry of buyers and sellers' preferences and induces some sophisticated traders to abstain from trading. However, one should be able to get equilibrium prices similar to those given by *Proposition 1* by having two block-monotonic bid distributions, one for buyers and one for sellers. This reasoning applies to any population of traders with heterogeneous

Multiunit demand and supply: results should continue to hold if every trader faces the same (finite) unit limit as a function of their bid price.[40] This means that a trader can trade more units if he bids a lower (higher) price, but that all traders face the same unit limit as a function of the bid. If the bidding strategies of sophisticated traders involve trading the maximum number of units allowed at each price, demand and supply will be measurable with respect to V. In turn, the market clearing price will be determined by V and the same conclusions regarding sophisticated bidding behavior should apply, leading to prices as those described by *Proposition 1*. A sophisticated trader facing such prices does not have an incentive to lower the number of units specified in her bid. Note that measurability breaks down if we have an heterogeneous distribution of individual (unit) endowments.[41]

Endogenous roles: in most asset markets agents can decide whether to buy or to sell, rather than being exogenously assigned to one side of the market. A way to introduce this choice in the model is to let a trader be a seller whenever her bid falls below the market price and become a buyer otherwise. It turns out that prices in this modified double auction constitute a special case of the original model. Notice that the lemmas regarding the bidding behavior of sophisticated traders still hold due to the symmetry of preferences: if a trader is happy with being a buyer for prices below b, she is also happy with being a seller for prices above b. Hence, her bidding behavior does not change compared to the case of exogenous roles. What changes is that now she always trades, so total trade volume at the market clearing price is equal to half the mass of units. Thus, $\rho(v, \eta)$ satisfies

$$0.5 \in [B_{\_}(\rho(v, \eta)|v, \eta), B(\rho(v, \eta)|v, \eta)],$$

which is the market clearing price in a symmetric market ($\gamma = 0.5$).

Unrestricted demand/supply schedules: if traders are allowed to submit a fully specified unit demand/supply schedule, we are back to a REE world, i.e. to the case in which agents decide whether to trade or not at each price. In general, allowing sophisticated traders to submit multiple bids increases the informational efficiency of prices by reducing the extent of mispricing. This suggests that a possible reason for prediction markets to perform well is that sophisticated traders are more *active* than naïve traders. To get an idea of how prices arise in this market, assume that naïve traders still submit a single bid and they are symmetrically distributed among buyers and sellers (i.e. the mass of naive sellers is $\eta\gamma$).[42] In this setting,

degrees of risk aversion, as long as the set of distinct attitudes toward risk is finite, so that the number of different block-monotonic distributions is also finite.

[40] This is true in many prediction markets, in which there is usually a limit on the dollar value of an order or, alternatively, a maximum order size.

[41] With heterogeneous endowments the mass of units for sale at or below a given price depends on the identity of traders bidding below the price and not only on the distribution of their signals.

[42] The rationale for this type of market is the following. If the cost of submitting an offer is

when $\rho(v,\eta) > v$ only sophisticated sellers wish to trade. Hence, supply is equal to $\eta\gamma H(\rho(v,\eta)|v) + (1-\eta)\gamma$ and demand is $\eta(1-\gamma)(1-H(\rho(v,\eta)|v))$. Then the market clearing price satisfies

$$H(\rho(v,\eta)|v) = 1 - \frac{\gamma}{\eta}. \tag{14}$$

For prices below values demand is $\eta(1-\gamma)(1-H(\rho(v,\eta)|v)) + (1-\eta)(1-\gamma)$ and supply is $\eta\gamma H(\rho(v,\eta)|v)$. Thus,

$$H(\rho(v,\eta)|v) = \frac{1-\gamma}{\eta}. \tag{15}$$

Notice that prices given by (14) are always lower than prices given by (15) for each v, since $\frac{1-\gamma}{\eta} > 1 - \frac{\gamma}{\eta}$. Thus, if (14) leads to $\rho(v,\eta) > v$ then (15) cannot lead to $\rho(v,\eta) < v$ and viceversa.

We have the following cases:

(i) For $\eta < \min\{\gamma, 1-\gamma\}$ prices equal values for all v, given that (14)-(15) cannot be satisfied. Thus, allowing multiple bids can substantially increase the lower bound $\underline{\eta}$ below which prices are equal values everywhere.[43]

(ii) For $\eta > \max\{\gamma, 1-\gamma\}$ the space of values can be divided into three regions: intervals in which prices are above values, intervals in which prices are below and intervals with prices equal to values. Furthermore, there exist $0 < v' < v'' < 1$ such that prices are above values for $v < v'$ and below values for $v > v''$ given that, by *Assumption 3*, (14) implies $\rho(0,\eta) > 0$ and (15) implies $\rho(1,\eta) < 1$.[44] Hence, a *global* favorite-longshot bias exists when there are many naïve traders. Figure 4 shows equilibrium prices using the same distributions F, G and H as in *Example 1*.

(iii) For $\eta \in [\min\{\gamma, 1-\gamma\}, \max\{\gamma, 1-\gamma\}]$ there is a combination of intervals without mispricing and intervals with $\rho(v,\eta) > (<)\ v$ when $\gamma > (<)\ 1-\gamma$, given that either (14) or (15) are never satisfied.

These possible extensions suggest that the model implications hold in more general large double auctions. A topic for future research would be to investigate whether these results represent a reasonable approximation of what happens in economies with a large but finite number of agents.

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zero and there are no restrictions on short sales, sophisticated traders could replicate their demand schedule by simultaneously submitting multiple bids and asks. On the other hand, naïve traders would lack the expertise to create complex bidding schemes.

[43] Recall that, in *Example 1*, $\underline{\eta} \approx 0.016$ when agents are restricted to one bid, whereas $\underline{\eta} = 0.5$ when they can place a continuum of bids (i.e. full demand/supply schedules).

[44] The intervals without mispricing happen when prices satisfying (14) are below while for those same values prices satisfying (15) are above them.

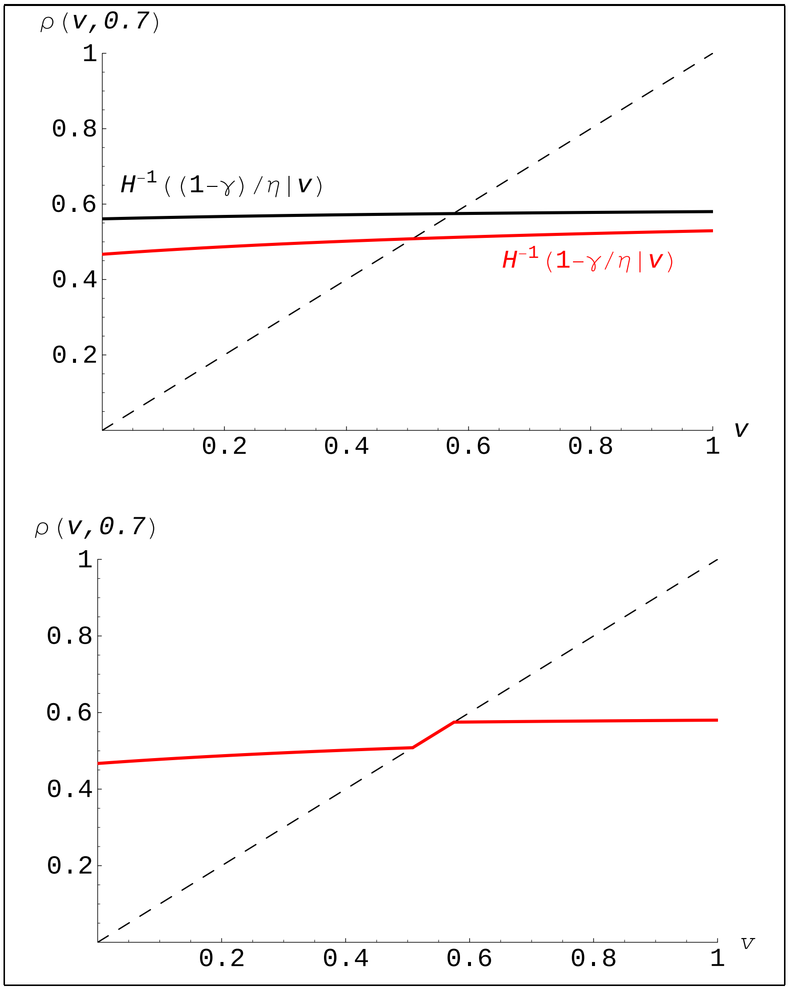

Figure 4: $\rho(v, \eta)$ satisfying (14)-(15) and equilibrium prices for $\gamma = 0.5$, $\eta = 0.7$

A Appendix: Proofs

A.1 Proofs of *Proposition 1* and *Corollary 1*

Proof of *Lemma 1*. Let $\rho(V, \eta)$ be the price function resulting from strategy profile $\beta(\cdot, \cdot)$, and assume buyer t and seller t' bid b when they receive signal s, i.e. $\beta(s, t) = \beta(s, t') = b$. If we subtract (5) from (4) we get

$$\pi^{buy}(s, t) = \pi^{sel}(s, t') + \mathbb{E}((V - \rho(V, \eta))|s). \tag{16}$$

Since the last term does not depend on b, a buyer and a seller receiving the same signal will have the same preference ranking over bids. ■

Let $\rho_+^{-1}(b,\eta) := \max\{v : \rho(v,\eta) = b\}$, $\rho_-^{-1}(b,\eta) := \min\{v : \rho(v,\eta) = b\}$.

Lemma 2 (Sophisticated Bidding in Monotone Equilibria (I)) *Let $\beta(\cdot,\cdot)$ be the sophisticated traders' strategy profile in a monotone equilibrium and (v_1, v_2) be a non-degenerate set of asset values.*

(i) If $\rho(v,\eta) < v$ for all $v \in (v_1, v_2)$ and there is a trader t with $\beta(s,t) < \rho(v_1,\eta)$ for some s, then there exists $v' \in (\rho_-^{-1}(\beta(s,t),\eta), v_1]$ such that $\rho(v,\eta) \geq v$ for all $v \in (\rho_-^{-1}(\beta(s,t),\eta), v')$, with strict inequality in a non-null subset.

(ii) If $\rho(v,\eta) > v$ for all $v \in (v_1, v_2)$ and there is a trader t with $\beta(s,t) > \rho(v_2,\eta)$ for some s, then there exists $v' \in [v_2, \rho_+^{-1}(\beta(s,t),\eta))$ such that $\rho(v,\eta) \leq v$ for all $v \in (v', \rho_+^{-1}(\beta(s,t),\eta))$, with strict inequality in a non-null subset.

Proof of *Lemma 2.* Part (i): assume that $\rho(v,\eta) \leq v$ holds for all $v \in (\rho_-^{-1}(\beta(s,t),\eta), v_1)$. Then, given that $\mathbb{E}((V - \rho(V,\eta))1_{\{\rho(V,\eta)<v_2\}}|s) > 0$ for all s, a buyer would strictly prefer to bid v_2 than $\beta(s,t)$. Since preferences are symmetric, a seller would also prefer to bid v_2. A symmetric argument applies to (ii). ■

The following fact is used in the proofs of *Lemma 3* and *Proposition 1.*

Fact 1 *Let Assumptions 1 and 2 be satisfied. If $\rho(v,\eta) > v$ for all $v \in (v_1, v_2)$ and $\rho(v,\eta) < v$ for all $v \in (v_2, v_3)$ with $\rho(\cdot,\eta)$ increasing, then for all $s \in (0,1)$,*

(i) If $\mathbb{E}((V - \rho(V,\eta))1_{\{V\in[v_1,v_3]\}}|s) \leq 0$, then $\mathbb{E}((V - \rho(V,\eta))1_{\{V\in[v_1,v_3]\}}|s') < 0$ for all $s' < s$;

(ii) If $\mathbb{E}((V - \rho(V,\eta))1_{\{V\in[v_1,v_3]\}}|s) \geq 0$, then $\mathbb{E}((V - \rho(V,\eta))1_{\{V\in[v_1,v_3]\}}|s') > 0$ for all $s' > s$.

Proof of *Fact 1.* Let $\mathbb{E}((V - \rho(V,\eta))1_{\{V\in[v_1,v_3]\}}|s) \leq 0$. Thus,

$$\frac{1}{f(s)}\int_{v_1}^{v_2}(\rho(V,\eta) - V)f(s|v)g(v)dv \geq \frac{1}{f(s)}\int_{v_2}^{v_3}(V - \rho(V,\eta))f(s|v)g(v)dv. \quad (17)$$

By the strict monotone likelihood ratio of F (*Assumption 2*) we have that for all $s' < s$ and all $v' \in [v_1, v_2)$ and $v \in [v_2, v_3)$, $\frac{f(s'|v')}{f(s|v')} > \frac{f(s'|v)}{f(s|v)}$. Given this and the above inequality, we have that

$$\int_{v_1}^{v_2}(\rho(V,\eta) - V)f(s|v)\frac{f(s'|v)}{f(s|v)}g(v)dv > \int_{v_2}^{v_3}(V - \rho(V,\eta))f(s|v)\frac{f(s'|v)}{f(s|v)}g(v)dv. \quad (18)$$

Given that $f(s') > 0$ for all s' by the full support of F and G (*Assumption 1*), (18) implies that $\mathbb{E}((\rho(V,\eta) - V)1_{\{V\in[v_1,v_2]\}}|s') > \mathbb{E}((V - \rho(V,\eta))1_{\{V\in[v_2,v_3]\}}|s')$. A symmetric argument applies to part (ii). ■

Lemma 3 (Sophisticated Bidding in Monotone Equilibria (II)) *The mass of sophisticated traders submitting bids in* $\{\rho(v,\eta) : v - \rho(v,\eta) \neq 0\}$ *is zero in a monotone equilibrium, except perhaps when there is a positive mass at* $\rho(0,\eta)$ *or at* $\rho(1,\eta)$*, and* $1-\gamma = B(\rho(0,\eta)|v)$ *for all* $v \in [0, \rho_+^{-1}(0,\eta)]$ *(complete rationing) or* $1-\gamma = B_{\_}(\rho(1,\eta)|v)$ *for all* $v \in [\rho_-^{-1}(1,\eta), 1]$ *(no rationing), respectively.*

Proof of *Lemma 3.* By *Lemma 1*, I only need to look at a buyer's incentives. The proof goes along the following lines. First, I show that no sophisticated buyer is best-responding by bidding in the interior of an interval of prices in which $\rho(v,\eta) \neq v$. Second, I show that if $\rho(\cdot,\eta)$ is constant in an interval of values (i.e. the distribution of prices has an atom) a sophisticated buyer will only bid at the atom if she gets the object with probability zero or one, depending on whether the expected value of $\rho(V,\eta) - V$ at the atom is positive or negative, respectively. Otherwise, she would bid slightly above or below to either avoid trading or being rationed. Finally, I prove that these conditions cannot be satisfied at an atom in the interior of the price range. Therefore, the only possibility left for a sophisticated buyer bidding in $\{\rho(v,\eta) : v - \rho(v,\eta) \neq 0\}$ is to bid at the boundaries, with the condition that she does not trade almost surely when she bids $\rho(0,\eta)$ and that she trades with probability one when bidding $\rho(1,\eta)$.

Assume that a buyer bids in an interval $(\rho(v_1,\eta), \rho(v_2,\eta))$ where $v > \rho(v,\eta)$ and $\rho(v,\eta)$ is strictly increasing a.e. in (v_1,v_2), i.e. there is no atom in (v_1,v_2). In this case, she prefers to bid v_2 to any bid $b \in (\rho(v_1,\eta), \rho(v_2,\eta))$, given that her payoff increases by $\mathbb{E}((V-\rho(V,\eta))1_{\{\rho(V,\eta)\in(b,\rho(v_2,\eta))\}}|s)$, which is strictly positive for all s. If, on the other hand, $v < \rho(v,\eta)$ in (v_1,v_2), a buyer would prefer to bid below $\rho(v_1,\eta)$ given that $\mathbb{E}((V-\rho(V,\eta))1_{\{\rho(V,\eta)\in(\rho(v_1,\eta),b)\}}|s) < 0$ for all s.

Now assume there is an atom at $b \in (0,1)$. If there is a positive mass of sophisticated bids at b, with $\rho(v,\eta) = b$ on some interval (v_1,v_2), a buyer with signal s might bid b under one of these conditions: (i) $\mathbb{E}((V-b)1_{\{\rho(V,\eta)=b\}}|s) = 0$; (ii) $\mathbb{E}((V-b)1_{\{\rho(V,\eta)=b\}}|s) > 0$ with $\lambda(b,v) = 1$ for all $v \in (v_1,v_2)$ (i.e. no rationing); (iii) $\mathbb{E}((V-b)1_{\{\rho(V,\eta)=b\}}|s) < 0$ with $\lambda(b,v) = 0$ for all $v \in (v_1,v_2)$ (i.e. no trading when $\rho(\cdot,\eta) = b$).

In case (i), she is indifferent between bidding slightly above or below b. However, *Fact 1* implies that there can be at most one signal satisfying (i).[45] Therefore the mass of bids at b due to (i) is zero. $\lambda(b,v) = 1$ in (ii), because she would rather bid above b if she gets the object with probability less than one. Finally, in (iii) she may bid at b only because the probability of getting the object is zero ($\lambda(b,v) = 0$). Since in each of the latter two cases $\lambda(b,\cdot)$ is required to be zero or one in the whole interval (v_1,v_2), there cannot be two traders bidding at b with distinct signals satisfying (ii) and (iii), respectively. Accordingly, either (ii) or (iii) holds for all the sophisticated bidders bidding b.

[45] For (i) to hold $v < b$ in the lower part of (v_1,v_2) and $v > b$ in the upper part of the interval, so that *Fact 1* applies.

Now I show that (ii) and (iii) can only happen when $b = \rho(1,\eta)$ and $b = \rho(0,\eta)$, respectively.

Assume (ii) is satisfied for all bidders bidding b and let $\underline{s}$ be the lowest signal associated to b. Accordingly, a trader receiving $\underline{s}$ bids optimally at b if

$$\mathbb{E}((V - \rho(V,\eta))1_{\{p \leq b\}}|\underline{s}) \geq 0, \tag{19}$$

and

$$\mathbb{E}((V - \rho(V,\eta))1_{\{p > b\}}|\underline{s}) \leq 0. \tag{20}$$

By *Lemma 2*, we can apply *Fact 1* to (19) and (20).[46] Hence, all sophisticated traders with signals above $\underline{s}$ will bid at or above b (assuming $\lambda(b,v) = 1$). Likewise, given (20) and the fact that $\mathbb{E}((V - b)1_{\{\rho(V,\eta)=b\}}|\underline{s}) \geq 0$, all sophisticated traders with $s < \underline{s}$ will bid strictly below b.

For $\lambda(b,v) = 1$ we need the mass of sellers bidding strictly less than b be equal to the mass of buyers bidding at b or above. That is, for all $v \in (v_1, v_2)$,

$$\gamma[\eta H(b|v) + (1-\eta)F(\underline{s}|v)] = (1-\gamma)[\eta(1 - H(b|v)) + (1-\eta)(1 - F(\underline{s}|v))]. \tag{21}$$

Given that $B\_(b|v) = \eta H(b|v) + (1-\eta)F(\underline{s}|v)$, the above expression is satisfied when $B\_(b|v) = 1 - \gamma$ for all $v \in (v_1, v_2)$.

Now assume that $b < \rho(1,\eta)$, i.e. $v_2 < 1$ and $\rho(v,\eta) > b$ for all $v > v_2$. For that to happen we need $B(b|v) < 1-\gamma$ for all $v > v_2$. But this implies, by the continuity of $B(b|v)$ and $B\_(b|v)$, that there exists $v' < v_2$ such that $B\_(b|v) < B(b|v) \leq 1 - \gamma$ for all $v \geq v'$, which contradicts that $\lambda(b,v) = 1$ for all $v \in (v_1, v_2)$. Hence, (ii) is only possible in equilibrium if $b = \rho(1,\eta)$ and $v_2 = 1$.

A symmetric argument applies when (iii) is satisfied for almost all sophisticated traders bidding at b. In this case, the mass of sellers bidding at or below b needs to be equal to the mass of buyers bidding strictly above b for $\lambda(b,v) = 0$. This requires that $B(b|v) = 1 - \gamma$ for all $v \in (v_1, v_2)$. By the continuity of $B(b|v)$ and $B\_(b|v)$, $b = \rho(0,\eta)$ and $v_1 = 0$, otherwise there would be a subset of (v_1, v_2) for which $1 - \gamma \leq B\_(b|v) < B(b|v)$, contradicting that $\lambda(b,v) = 0$ for all $v \in (v_1, v_2)$.

Finally, if $\rho(1,\eta)$ is not an atom, the probability of rationing at $\rho(1,\eta)$ is zero and a buyer bidding $\rho(1,\eta)$ will always trade. In this case there can exist a positive mass of sophisticated bids at $\rho(1,\eta) < 1$, since any buyer bidding $\rho(1,\eta) < 1$ is indifferent between any two bids in $[\rho(1,\eta), 1]$. A symmetric argument can be made for bids at $\rho(0,\eta) > 0$. ■

Lemma 3 allows for the possibility of having sophisticated bids placed at an atom, at $\rho(0,\eta)$ or at $\rho(1,\eta)$, of the price distribution if either sellers or buyers bidding at the atom trade with probability one, respectively. However, as the next lemma shows, atoms can only occur for very particular naïve share and bid distributions.

[46] By the linearity of expectations, the conclusions of *Fact 1* also apply to a succession of intervals satisfying the conditions in the lemma.

Lemma 4 (No Atoms) *In any monotone equilibrium if there exists $v_1 < v_2$ such that $\rho(v,\eta) = b$ on (v_1, v_2) then*

(a) $\mathbb{E}((V - \rho(v,\eta))1_{\{V<v_2\}}|s) \geq 0$ *for all* s*, and* $H(\rho(v,\eta)|v) = \frac{1-\gamma}{\eta}$ *for all* $v \leq v_2$*;*

(b) $\mathbb{E}((V - \rho(v,\eta))1_{\{V<v_2\}}|s) \leq 0$ *for all* s*, and* $H(\rho(v,\eta)|v) = \frac{\eta-\gamma}{\eta}$ *for all* $v \geq v_1$*.*

Lemma 4 basically states that atoms in the price distribution are solely created by naïve traders, and that very special circumstances need to occur: the share of naïve bids is very high compared to γ (or to $1 - \gamma$); naïve traders completely determine prices at the low (high) end of the price range, with those prices being low (high) enough so that they do not encourage sophisticated traders to bid below (above) the atom; and the distribution of naïve bids is independent of asset values in the interval of values associated with the atom.[47]

Proof of *Lemma 4*. Assume there is an interval (v_1, v_2) such that $\rho(v,\eta) = b$ for all $v \in (v_1, v_2)$. By *Lemma 2* and *Fact 1*, if there exists a trader with signal s bidding below (above) b then it is optimal for all traders with signals below (above) s to also bid below (above) b. Accordingly, let $\underline{s} \in [0,1]$ be the highest signal associated with bids below b, and $\overline{s} \geq \underline{s}$ the lowest signal associated with bids above b. Since the distribution of naïve bids is atomless (*Assumption 3*), this implies that

$$B\_(b|v) = \eta H(b|v) + (1-\eta)F(\underline{s}|v),$$

and

$$B(b|v) = \eta H(b|v) + (1-\eta)F(\overline{s}|v).$$

There are two possible cases, depending on whether a positive mass of sophisticated bids is placed at b or not, i.e. whether $\underline{s} < \overline{s}$ or $\underline{s} = \overline{s}$.

If there is no positive mass of sophisticated bids at b, we have that $B(b|v) = B\_(b|v) = 1 - \gamma$ for all $v \in (v_1, v_2)$. Since $F(s|v)$ is strictly decreasing in v for all $s \in (0,1)$ and $H(b|v)$ is non-increasing in v for all $b \in [0,1]$, $B\_(b|v) = 1 - \gamma$ for all $v \in (v_1, v_2)$ only if $\underline{s} = 0$ or $\underline{s} = 1$.

a) $\underline{s} = 0$: in this case $H(b|v) = \frac{1-\gamma}{\eta}$ for all $v \in (v_1, v_2)$. But then, we need $\mathbb{E}((V - \rho(V,\eta))1_{\{\rho(V,\eta)\leq b\}}|s) = \mathbb{E}((V - \rho(V,\eta))1_{\{V<v_2\}}|s) \geq 0$ for all s, otherwise some sophisticated traders would rather bid below b. Finally, prices below b are completely determined by naïve bids, since no sophisticated trader bids below b, i.e. $H(\rho(v,\eta)|v) = \frac{1-\gamma}{\eta}$ for all $v \leq v_1$.[48]

[47] An example of equilibrium prices being constant in some interval of values is given by a high enough presence of random traders bidding uniformly in $[0,1]$. In this case, $H(b|v) = b$ for all v. Hence, if η is high enough so that $\mathbb{E}(V|0) \geq \frac{1-\gamma}{\eta}$, then $\rho(v,\eta) = \frac{1-\gamma}{\eta}$ for all v, with all sophisticated traders bidding at or above $\frac{1-\gamma}{\eta}$. In this case, all sophisticated buyers and no sophisticated seller engage in trade.

[48] This is possible in principle given that $H(\cdot|\cdot)$ is increasing in its first argument and decreasing in its second argument.

b) $\underline{s} = 1$: in this case $H(b|v) = \frac{\eta-\gamma}{\eta}$ for all $v \in (v_1, v_2)$. In addition, we need $\mathbb{E}((V - \rho(V,\eta))1_{\{V<v_2\}}|s) \leq 0$ for all s. Since no sophisticated trader bids above b, prices above b are given by $H(\rho(v,\eta)|v) = \frac{\eta-\gamma}{\eta}$ for all $v \geq v_2$.

If there is a positive mass of sophisticated bids at b, *Lemma 3* applies, requiring either that $B\_(b|v) = 1-\gamma$ or $B(b|v) = 1-\gamma$. The former requires $\underline{s} = 0$ or $\underline{s} = 1$, while the latter can be possible only if $\overline{s} = 0$ or $\overline{s} = 1$. Therefore, they reduce to the same conditions on $H(\cdot|\cdot)$ and $\mathbb{E}((V - \rho(V,\eta))1_{\{\rho(V,\eta)\leq b\}}|s)$. ■

Proof of *Proposition 1.* By the monotonicity of $\rho(\cdot,\eta)$ and *Lemma 3*, all the mass of sophisticated bids in $(\rho(0,\eta), \rho(1,\eta))$ is placed in a countable collection of disjoint intervals in which $\rho(v,\eta) = v$. Let $\mathcal{V}$ be the complement of such set in $[0,1]$. Thus, $\mathcal{V} \supseteq \{v : \rho(v,\eta) \neq v\}$ by *Lemma 3*. Assume $\mathcal{V}$ is non-empty, otherwise *Proposition 1* holds trivially.

Denote $B^*(\cdot|\cdot)$ the cdf of sophisticated bids and assume that $B^*(\cdot|\cdot)$ is atomless.[49] Accordingly, $B^*_-(\cdot|v) = B^*(\cdot|v)$ for all $v \in [\rho(0,\eta), \rho(1,\eta)]$ and $\mathcal{V}$ can be expressed, without loss of generality, as the countable union of disjoint closed intervals $[\underline{v}_k, \overline{v}_k]$, such that $\rho(v,\eta)$ is given by

$$H(\rho(v,\eta)|v) = \frac{1-\gamma-(1-\eta)B^*(\rho(\underline{v}_k,\eta)|v)}{\eta} \text{ for all } v \in [\underline{v}_k, \overline{v}_k]. \tag{22}$$

Further assume that prices are not a.e. equal to values in $[\underline{v}_k, \overline{v}_k]$. Otherwise, redefine $\mathcal{V}$ not to include such interval.

Notice that $B^*(\rho(\underline{v}_k,\eta)|v) = B^*(\rho(\overline{v}_k,\eta)|v)$ by *Lemma* 3 for all k, including mispricing intervals with $\underline{v}_1 = 0$ (i.e. when $\rho(0,\eta) > 0$) and $\overline{v}_K = 1$ ($\rho(1,\eta) < 1$). Hence, we just need to show that $B^*(\rho(\underline{v}_k,\eta)|v) = F(s^*_k|v)$, for all k and all $v \in [\underline{v}_k, \overline{v}_k]$, with s^*_k satisfying (7)-(9).

By *Lemma 2* and the fact that $\rho(0,\eta) > 0$ and $\rho(1,\eta) < 1$, there exist v'_k, v''_k with $\underline{v}_k < v'_k \leq v''_k < \overline{v}_k$ such that $\rho(v,\eta) \geq v$ a.e. in $[\underline{v}_k, v'_k]$ with strict inequality in a non-null subset, and $\rho(v,\eta) \leq v$ a.e. in $[v''_k, \overline{v}_k]$ with strict inequality in a non-null subset.[50] Given this, if bidding in $[\overline{v}_{k-1}, \rho(\underline{v}_k)]$ is optimal for a seller with signal s, then $\mathbb{E}((V-\rho(V,\eta))1_{\{V\in(\underline{v}_k,\overline{v}_k)\}}|s) + \sum_{k'>k}\mathbb{E}((V-\rho(V,\eta))1_{\{V\in(\underline{v}'_k,\overline{v}'_k)\}}|s) \leq 0$ with $\mathbb{E}((V-\rho(V,\eta))1_{\{V\in(\underline{v}_k,\overline{v}_k)\}}|s) \leq 0$, otherwise she would bid above $\rho(\overline{v}_k)$. But these inequalities hold strictly for all $s' < s$ by *Fact 1*. Hence, bidding above $\rho(\overline{v}_k)$ is strictly dominated by bidding in $[\overline{v}_{k-1}, \rho(\underline{v}_k)]$ for all sellers with $s' < s$. A symmetric argument can be used for all $s' > s$ when it is optimal for a seller with signal s to bid in $[\rho(\overline{v}_k), \underline{v}_{k+1}]$. By *Lemma 1* the same applies for a buyer. Therefore,

[49] This implies that $\rho(0,\eta) > 0$ and $\rho(1,\eta) < 1$, given that $H(0) = 0 < 1-\gamma$ and $H(1) = 1 > 1-\gamma$ by *Assumption 3*.

[50] In what follows, I use the convention, $\overline{v}_0 = 0$ and $\underline{v}_{K+1} = 1$.

$B^*(\rho(\underline{v}_k)|v) = F(s_k^*|v)$ for some signal $s_k^* > 0$. Moreover, s_k^* needs to satisfy (7) if $\underline{v}_k > 0$ and (8) whenever $\overline{v}_k < 1$, given that $\rho(v, \eta) = v$ in $(\overline{v}_{k-1}, \underline{v}_k)$ and in $(\overline{v}_k, \underline{v}_{k+1})$ and that $H(\cdot|\cdot)$, $F(\cdot|\cdot)$ are atomless distributions. Finally, condition (9) is just the equilibrium condition for a seller with $s \leq s_k^*$ ($s > s_k^*$) to optimally bid below $\rho(\underline{v}_k)$ (above $\rho(\overline{v}_k)$), which also implies that $s_{k-1}^* < s_k^*$ for all $k > 1$.

Now assume that $B^*(\cdot|\cdot)$ has an atom. Since $H(\cdot|\cdot)$ does not have atoms in $(\rho(0, \eta), \rho(1, \eta))$, neither can $B^*(\cdot|\cdot)$. An atom of $B^*(\cdot|\cdot)$ at $b \in (\rho(0, \eta), \rho(1, \eta))$ would imply that $B_{\_}(b|b) < B(b|b)$, creating an atom of the price distribution at b, which leads to a mispricing interval where sophisticated bids are placed, a contradiction of *Lemma 3*. Therefore, $B^*(\cdot|\cdot)$ can have an atom only in $\{\rho(0, \eta), \rho(1, \eta)\}$.

If there is an atom in $B^*(\cdot|\cdot)$ at $\rho(0, \eta)$, the price distribution may or may not have an atom at $\rho(0, \eta)$. If the price distribution has an atom at $\rho(0, \eta)$, by *Lemma 4*, $B^*(\rho(0, \eta)|v) = 1$ for all v and $H(\rho(0, \eta)|v) = \frac{\eta - \gamma}{\eta}$ for all $v \leq \rho_+^{-1}(\rho(0, \eta), \eta)$. Accordingly, $B^*(\rho(0, \eta)|v) = F(1|v)$ and (6) is satisfied. If the price distribution does not have an atom at $\rho(0, \eta)$, $\rho(0, \eta)$ is given by (22), i.e.

$$H(\rho(0, \eta)|0) = \frac{1 - \gamma - (1 - \eta)B^*(\rho(0, \eta)|0)}{\eta}. \tag{23}$$

Hence, if a mispricing interval starts at $\rho(0, \eta)$ (i.e. $\underline{v}_1 = 0$), *Lemma 2* applies to the interval $[0, \overline{v}_1]$ and, by *Fact 1*, there exists a signal $s_1^* > 0$ satisfying (9) such that $B^*(\rho(0, \eta)|v) = F(s_1^*|v)$.

Finally, assume $B^*(\cdot|\cdot)$ has an atom at $\rho(1, \eta)$. If the price distribution has an atom at $\rho(1, \eta)$, $B_{\_}^*(\rho(1, \eta)|v) = 0$ for all v and $H(\rho(1, \eta)|v) = \frac{1-\gamma}{\eta}$ for all $v \geq \rho_{\_}^{-1}(\rho(0, \eta))$ by *Lemma 4*. Thus, $B_{\_}^*(\rho(1, \eta)|v) = F(0|v)$ and (6) is also satisfied. If the price distribution does not have an atom at $\rho(1, \eta)$, $\rho(1, \eta)$ is given by

$$H(\rho(1, \eta)|1) = \frac{1 - \gamma - (1 - \eta)B_{\_}^*(\rho(1, \eta)|1)}{\eta}. \tag{24}$$

Therefore, if a mispricing interval ends at $\rho(1, \eta)$, *Lemma 2* applies to the interval $[\overline{v}_K, 1]$ and, by *Fact 1*, there exists a signal $s_K^* < 1$ satisfying (9) such that $B^*(\rho(0, \eta)|v) = F(s_1^*|v)$. ■

Proof of *Corollary 1*. First, notice that by *Lemma 2*, all price intervals $(\rho(\underline{v}_k, \eta), \rho(\overline{v}_k, \eta))$ with sophisticated bids below and above them satisfy the first part of *Corollary 1*, i.e. prices are above values in the lower portion of the interval $(\underline{v}_k, \overline{v}_k)$ and below values in the upper portion. When there are no sophisticated bids placed below $(\rho(\underline{v}_k, \eta), \rho(\overline{v}_k, \eta))$ we have that $\underline{v}_k = 0$ given that $H(0) = 0 < 1 - \gamma$ implies $\rho(0, \eta) > 0$. If there are sophisticated bids above this interval part (ii) of *Lemma 2* applies. Finally, when there are no sophisticated bids above $(\rho(\underline{v}_k, \eta), \rho(\overline{v}_k, \eta))$ we have that $\overline{v}_k = 1$ and $\rho(\overline{v}_k, \eta) < 1$. In addition, if there are sophisticated bids below $(\rho(\underline{v}_k, \eta), \rho(\overline{v}_k, \eta))$, part (i) of *Lemma 2* applies. Hence, these two cases also satisfy the first part of the corollary. The same reasoning applies when there is an atom at $(\underline{v}_k, \overline{v}_k)$. Since atoms can only happen when there are no bids either below or above the atom (*Lemma 4*), we are in one of the above cases.

Regarding the second part of *Corollary 1*, assume there is a mispricing interval $(\underline{v}_k, \overline{v}_k)$ that can be partitioned into two subintervals, $(\underline{v}_k, \hat{v}]$ and $(\hat{v}, \overline{v}_k)$, such that, in each of them, prices are above values in the lower portion of the subinterval and below values in the upper portion with $\mathbb{E}((V - \rho(V, \eta))1_{\{V \in (\hat{v}, \overline{v}_k)\}} | s_k^*) < 0$ with $s_k^* < 1$. In such case, a sophisticated seller bidding just above $\rho(\overline{v}_k, \eta)$ with signal close to s_k^* would rather deviate and bid $\hat{v}$ and get higher payoffs by trading when $v \in (\overline{v}_k, \hat{v}]$. ■

A.2 Proof of *Proposition 2*

I provide a series of technical lemmas and facts related to $\alpha(\cdot, \eta)$, which are used in the proof of *Proposition 2*. They show that $\alpha(\cdot, \eta)$ is increasing for small values of η, non-monotonic for intermediate values and not well-defined and decreasing when η is high enough. Prices must equal values everywhere in the first case, and cannot equal values in areas where $\alpha(\cdot, \eta)$ is either not defined or decreasing.

In what follows, D_i is the partial derivative with respect to the ith argument.

Lemma 5 *If Assumptions 1-4 are satisfied the following statements are true:*

(i) $\alpha(0, \eta)$ *is well-defined for* $\eta < \gamma$*, strictly positive and increasing in* η*;* $\alpha(1, \eta)$ *is well-defined for* $\eta < 1 - \gamma$*, strictly less than one and decreasing in* η*.*

(ii) If $D_1\alpha(v, \eta) \leq 0$ *then* $D_1\alpha(v, \eta') < 0$ *for all* $\eta' > \eta$ *for which* $\alpha(v, \eta)$ *is well defined.*

(iii) There exists $\underline{\eta} \in (0, \min\{\gamma, 1 - \gamma\})$ *such that* $\alpha(\cdot, \eta)$ *is well-defined and non-decreasing for all* $\eta \leq \underline{\eta}$*, and it is non-monotonic or decreasing for all* $\eta > \underline{\eta}$ *in the subset of values where it is well-defined.*

(iv) If $H'(v) > 0$ *for all* v *such that* $H(v) = 1 - \gamma$*, there exists* $\overline{\eta} \in [\underline{\eta}, 1)$ *such that* $\alpha(\cdot, \eta)$ *is decreasing whenever it is well-defined for all* $\eta > \overline{\eta}$*.*

Proof of Lemma 5.
<u>Part (i)</u>: since $H(0) = 0$, $\alpha(0, \eta) = F^{-1}(\frac{1-\gamma}{1-\eta}|0)$, which is well-defined if $\eta < \gamma$. Since $F(\cdot|v)$ has full support for all v and $\frac{1-\gamma}{1-\eta}$ is increasing in η, $\alpha(0, \eta)$ is increasing in η. Similarly, $H(1) = 1$ so $s(1, \eta) = F^{-1}(\frac{1-\gamma-\eta}{1-\eta}|1)$ is well-defined for $\eta < 1 - \gamma$ and decreasing in η.

<u>Part (ii)</u>: by the a.e. smoothness of H and F (*Assumptions 1* and *3*), $\alpha(v, \eta)$ is a.e. differentiable. I differentiate both sides of (10) to obtain $D_1\alpha(v, \eta)$:

$$D_1\alpha(v,\eta) = -\frac{\frac{\eta}{1-\eta}H'(v) + D_2F(\alpha(v,\eta)|v)}{f(\alpha(v,\eta)|v)}. \tag{25}$$

Note that $f(\cdot|\cdot) > 0$ by the full support assumption and $D_2F(\cdot|\cdot) < 0$ by strict MLRP. Therefore, for $D_1\alpha(v,\eta) \leq 0$ we need the numerator of (25) to be non-positive, i.e.

$$\frac{\eta}{1-\eta}H'(v) + D_2F(\alpha(v,\eta)|v) \geq 0. \tag{26}$$

Thus, if we show that (26) implies

$$\frac{\partial}{\partial\eta}\left[\frac{\eta}{1-\eta}H'(v) + D_2F(\alpha(v,\eta)|v)\right] > 0,$$

which is equivalent to

$$\frac{H'(v)}{(1-\eta)^2} > -D_2f(\alpha(v,\eta)|v)D_2\alpha(v,\eta), \tag{27}$$

then we would have shown that if the numerator of (25) is negative, it becomes more negative as η grows. This will suffice to prove part (ii) of the lemma.

Given that $D_2\alpha(v,\eta) = \frac{1-\gamma-H(v)}{(1-\eta)^2 f(\alpha(v,\eta)|v)}$, (27) can be expressed as

$$H'(v) > -(1-\gamma-H(v))\frac{D_2f(\alpha(v,\eta)|v)}{f(\alpha(v,\eta)|v)}. \tag{28}$$

Therefore, we need to prove that (26) implies (28). There are two possible cases: $H(v) < 1-\gamma$ and $H(v) > 1-\gamma$.[51] But before considering them, notice that the strict MLRP of f implies that $\frac{D_2f(s|v)}{f(s|v)} \in \left[\frac{D_2F(s|v)}{F(s|v)}, \frac{-D_2F(s|v)}{1-F(s|v)}\right]$ for all $s \in (0,1)$ and all v.[52] These bounds come from the fact that $\frac{F(s|v)}{f(s|v)}$ is decreasing in v and $\frac{1-F(s|v)}{f(s|v)}$ is increasing in v for all s, i.e.

$$\frac{\partial}{\partial v}\left[\frac{F(s|v)}{f(s|v)}\right] = \frac{f(s|v)D_2F(s|v) - D_2f(s|v)F(s|v)}{f^2(s|v)} \leq 0, \tag{29}$$

and

$$\frac{\partial}{\partial v}\left[\frac{1-F(s|v)}{f(s|v)}\right] = \frac{-f(s|v)D_2F(s|v) - D_2f(s|v)(1-F(s|v))}{f^2(s|v)} \geq 0. \tag{30}$$

<u>Case 1</u>: $H(v) < 1-\gamma$. If we divide both sides of (26) by $F(\alpha(v,\eta)|v)$,[53] we obtain

[51] If $H(v) = 1-\gamma$, (28) is satisfied given that $H'(v) > 0$ is needed for (26) to hold.
[52] These bounds are well defined since $F(s|v) \in (0,1)$ for all $s \in (0,1)$ by *Assumption 1*.
[53] $F(\alpha(v,\eta)|v) > 0$ whenever $H(v) < 1-\gamma$: if $F(\alpha(v,\eta)|v) = 0$, then $1-\gamma = \eta H(v) \leq H(v)$.

$$\frac{\eta}{1-\eta}\frac{H'(v)}{F(\alpha(v,\eta)|v)} \geq -\frac{D_2F(\alpha(v,\eta)|v)}{F(\alpha(v,\eta)|v)} \geq -\frac{D_2f(\alpha(v,\eta)|v)}{f(\alpha(v,\eta)|v)}.$$

Substituting $F(\alpha(v,\eta)|v) = \frac{1-\gamma-\eta H(v)}{1-\eta}$ in the above expression and multiplying both sides by $1-\gamma-H(v)$ we get

$$H'(v)\frac{\eta(1-\gamma-H(v))}{1-\gamma-\eta H(v)} \geq -(1-\gamma-H(v))\frac{D_2f(\alpha(v,\eta)|v)}{f(\alpha(v,\eta)|v)}. \tag{31}$$

Since $H(v) < 1-\gamma$, $\gamma \in (0,1)$ and $\eta \in (0,1)$,[54] $\frac{\eta(1-\gamma-H(v))}{1-\gamma-\eta H(v)}$ is strictly positive and less than one. Hence, (31) implies (28) given that $H'(v) > 0$ by (26).

Case 2: $H(v) > 1-\gamma$. Two subcases need to be considered. If $D_2f(\alpha(v,\eta)|v) \leq 0$ the right-hand side of (28) is non-positive. Thus, (28) is satisfied for all v such that $H'(v) > 0$ and all η. When $D_2f(\alpha(v,\eta)|v) > 0$, dividing both sides of (26) by $1-F(\alpha(v,\eta)|v)$ leads to[55]

$$\frac{\eta}{1-\eta}\frac{H'(v)}{1-F(\alpha(v,\eta)|v)} \geq -\frac{D_2F(\alpha(v,\eta)|v)}{1-F(\alpha(v,\eta)|v)} \geq \frac{D_2f(\alpha(v,\eta)|v)}{f(\alpha(v,\eta)|v)}.$$

Substituting $F(\alpha(v,\eta)|v)$ and rearranging terms, the above inequality becomes

$$\begin{aligned}
\eta H'(v) &\geq (1-\eta+\eta H(v)-(1-\gamma))\frac{D_2f(\alpha(v,\eta)|v)}{f(\alpha(v,\eta)|v)} \\
&\geq -(1-\gamma-H(v))\frac{D_2f(\alpha(v,\eta)|v)}{f(\alpha(v,\eta)|v)}.
\end{aligned} \tag{32}$$

The last inequality implies (28) and holds because $1-\eta+\eta H(v) \geq H(v)$.

Part (iii): Before proving this part, note that $\alpha(\cdot,\eta)$ is well-defined in $[0,1]$ iff $\eta \leq \eta_\gamma := \min\{\gamma, 1-\gamma\}$, given that $\frac{1-\gamma-\eta H(v)}{1-\eta} \in [\frac{1-\gamma-\eta}{1-\eta}, \frac{1-\gamma}{1-\eta}]$ for all $v \in [0,1]$.

First, I show that there exists $\underline{\eta}_l > 0$ such that $\alpha(\cdot,\eta)$ is non-decreasing in $[0,1]$ for all $\eta < \underline{\eta}_l$. Since $F(s|v)$ is increasing in s and decreasing in v, we have that, for $\eta \leq \eta_\gamma$ and all $v \in [0,1]$, $\alpha(v,\eta) \in [F^{-1}(\frac{1-\gamma-\eta}{1-\eta}|0), F^{-1}(\frac{1-\gamma}{1-\eta}|1)] \subset (0,1)$. This implies that $D_2F(\alpha(v,\eta)|v) < 0$ for all $v \in (0,1)$. If $v \leq \underline{b}^H$ then $\alpha(v,\eta) = F^{-1}(\frac{1-\gamma}{1-\eta}|v)$ which is strictly increasing in v for $\eta \leq \eta_\gamma$. Similarly, when $v \geq \overline{b}^H$ we have that $\alpha(v,\eta) = F^{-1}(\frac{1-\gamma-\eta}{1-\eta}|v)$ is strictly increasing. Finally, notice that $D_1\alpha(v,\eta) \to \frac{-D_2F(\alpha(v,\eta)|v)}{f(\alpha(v,\eta)|v)} > 0$ as $\eta \to 0$ for all $v \in (\underline{b}^H, \overline{b}^H)$. By the continuity of D_2F, f

[54] If (26) holds then $\eta > 0$. Also, $\alpha(\cdot,1)$ is not well defined.

[55] $F(\alpha(v,\eta)|v) < 1$ whenever $H(v) > 1-\gamma$: if $F(\alpha(v,\eta)|v) = 1$ then $1-\gamma = \eta H(v)+1-\eta \geq H(v)$.

(*Assumption 1*) and $\alpha(.,\eta)$, we can find $\underline{\eta}_l > 0$ such that $\alpha(v,\eta)$ is non-decreasing for all $\eta \leq \underline{\eta}_l$.

Next, I show that there exists $\underline{\eta}_u < \eta_\gamma$ such that $\alpha(v,\eta)$ is strictly decreasing for some v for all $\eta > \underline{\eta}_u$. By the continuity of $\alpha(.,.)$, the existence of $\underline{\eta}_u$ is implied by the fact that, for all $\eta \geq \eta_\gamma$, there is a non-null subset of values in which $\alpha(.,\eta)$ is strictly decreasing:

1. If $\eta \geq \gamma$, then $\frac{1-\gamma}{1-\eta} \geq 1$. Since $\frac{1-\gamma-\eta}{1-\eta} < 1$ and $H(\cdot)$ is continuous, there exists an interval of values $[\underline{v},\overline{v}]$ such that $\frac{1-\gamma-\eta H(\underline{v})}{1-\eta} = 1$ and $\frac{1-\gamma-\eta H(v)}{1-\eta}$ is strictly decreasing in v for all $v \in [\underline{v},\overline{v}]$. Hence, by the full support assumption, $\alpha(\underline{v},\eta) = 1$ and $\alpha(v,\eta) < 1$ for all $v \in (\underline{v},\overline{v}]$, which implies that $D_1\alpha(v,\eta) < 0$ for some subset of $(\underline{v},\overline{v}]$.

2. If $\eta \geq 1-\gamma$, we have that $\frac{1-\gamma-\eta}{1-\eta} \leq 0$. Since $\frac{1-\gamma}{1-\eta} > 0$, there exists an interval of values $[\underline{v}',\overline{v}']$ such that $\frac{1-\gamma-\eta H(\overline{v}')}{1-\eta} = 0$ and $\frac{1-\gamma-\eta H(v)}{1-\eta}$ is strictly decreasing in v for all $v \in [\underline{v}',\overline{v}']$. Hence, $\alpha(\overline{v}',\eta) = 0$ and $\alpha(v,\eta) > 0$ for all $v \in [\underline{v}',\overline{v}')$, which implies that $D_1\alpha(v,\eta) < 0$ or some subset of $[\underline{v}',\overline{v}')$.

Finally, it remains to be shown that $\underline{\eta}_l = \underline{\eta}_u = \underline{\eta}$. By part (ii) of the lemma, if $D_1\alpha(v;\eta) \leq 0$ then $D_1\alpha(v;\eta') < 0$ for all $\eta' \in (\eta,\eta_\gamma)$. But this also implies that if $D_1\alpha(v,\eta) \geq 0$, then $D_1\alpha(v,\eta'') > 0$ for all $\eta'' < \eta$. Therefore, given that $D_1\alpha(v;\cdot)$ is well-defined and continuous in $[0,\eta_\gamma]$ for all v, $\underline{\eta}_l = \underline{\eta}_u = \underline{\eta}$ and is given by the highest η such that $D_1\alpha(v,\eta) \geq 0$ for all v, i.e.

$$\underline{\eta} := \sup_{\eta} \left\{ \eta \in (0,1) : \eta \leq -\frac{D_2F(F^{-1}(\frac{1-\gamma-\eta H(v)}{1-\eta}|v)|v)}{H'(v) - D_2F(F^{-1}(\frac{1-\gamma-\eta H(v)}{1-\eta}|v)|v)} \;\; \forall v \right\}. \qquad (33)$$

Part (iv): note that, as $\eta \to 1$, $\frac{1-\gamma-\eta H(v)}{1-\gamma} \to \infty$ for all v such that $H(v) < 1-\gamma$ and $\frac{1-\gamma-\eta H(v)}{1-\gamma} \to -\infty$ for all v such that $H(v) > 1-\gamma$. Therefore, for high enough η, $\alpha(\cdot,\eta)$ is only well-defined in a small neighborhood of all v such that $H(v) = 1-\gamma$. If for any such v we have that $H'(v) > 0$ then $\alpha(\cdot,\eta)$ will be decreasing in such neighborhood.[56] By part (ii) of the lemma, if $\alpha(\cdot,\eta)$ is decreasing, it is decreasing for all $\eta' > \eta$. ■

[56]In this case there is a unique v such that $H(v) = 1-\gamma$. Assume otherwise that there are two such values v, v' such that $H'(v), H'(v') > 0$. Since $H(0) = 0$ and $H(1) = 1$, by the continuity of $H(\cdot)$, there would have to be a value $v'' \in$ such that $H(v'') = 1-\gamma$ and $H'(v'') < 0$.

Fact 2 *Let $\mathcal{S}$ be a measurable subset of $[0,1]$ and $s \in (0,1)$ be such that $\mathbb{P}(\mathcal{S}|v) = F(s|v)$ for some $v \in [0,1)$. Then, $D_2\mathbb{P}(\mathcal{S}|v) \geq D_2F(s|v)$.*

Proof of Fact 2. Assume $\mathcal{S} \cap [s,1]$ is a non-null set, otherwise $\mathbb{P}(\mathcal{S}|v) = F(s|v)$ for all v by the full support of $F(\cdot|v)$ for all v. Since $\mathbb{P}(\mathcal{S}|v) = F(s|v) = \mathbb{P}([0,s]|v)$, we have that

$$\mathbb{P}([s,1] \cap \mathcal{S}|v) = \mathbb{P}([0,s] \setminus \mathcal{S}|v).$$

By the strict MLRP of $F(\cdot|\cdot)$, the left hand side is strictly greater than the right-hand side for all $v' > v$. Thus, $D_2\big[\mathbb{P}(\mathcal{S}|v) - \mathbb{P}([0,s]|v)\big] \geq 0$. ■

Lemma 6 *If $\alpha(\cdot,\eta)$ is strictly decreasing in some interval $[v_1, v_2]$ then any monotone equilibrium price $\rho(\cdot,\eta)$ satisfies $\rho(v,\eta) \neq v$ a.e. in $[v_1, v_2]$.*

Proof of Lemma 6. Assume $\rho(v,\eta) = v$ and $\rho(v',\eta) = v'$ for some $v' > v$ with $\alpha(v,\eta) > \alpha(v',\eta)$. Accordingly, if the mass of sophisticated bids below v is given by bidders with signals in $[0,\alpha(v,\eta)]$, then the mass of bids below $v' > v$ is strictly smaller than the mass of bids below v, a contradiction. Hence, it must be that there is an alternative, well-defined distribution of sophisticated bids $B^a(\cdot|\cdot)$ such that $B^a(v|v) = \frac{1-\gamma-\eta H(v)}{1-\eta}$ $(= F(\alpha(v,\eta)|v))$ for all $v \in [v_1, v_2]$. Since $\alpha(\cdot,\eta)$ is decreasing in that interval, we have that $\frac{d}{dv}B^a(v|v) = -\frac{\eta}{1-\eta}H'(v) < D_2F(\alpha(v,\eta)|v)$ (see (26)).

Denoting $\beta^a(s,t)$ the bid of sophisticated trader t when she receives signal s, we have that

$$B^a(v|v) = \int_{\mathcal{T}}\int_0^1 1_{\{\beta^a(s,t)\leq v\}} f(s|v)dsd\mu = \int_{\mathcal{T}} \mathbb{P}(\mathcal{S}^a(v,t)|v)d\mu,$$

where $\mathcal{S}^a(v,t) = \{s \in [0,1] : \beta^a(s,t) \leq v\}$.

By *Fact 2*, $D_2\mathbb{P}(\mathcal{S}^a(v,t)|v) \geq D_2F(s^a(v,t)|v)$ with $s^a(v,t)$ being the signal such that $\mathbb{P}(\mathcal{S}^a(v,t)|v) = F(s^a(v,t)|v)$. Accordingly, given that $D_1B^a(v|v) \geq 0$,

$$\frac{d}{dv}B^a(v|v) = D_1B^a(v|v) + D_2B^a(v|v) \geq \int_{\mathcal{T}} D_2F(s^a(v,t)|v)d\mu.$$

Therefore, it is enough to show that $\int_{\mathcal{T}} D_2F(s^a(v,t)|v)d\mu \geq D_2F(\alpha(v,\eta)|v)$ whenever $\int_{\mathcal{T}} F(s^a(v,t)|v)d\mu = F(\alpha(v,\eta)|v)$ in order to prove that there is no $B^a(\cdot|\cdot)$ leading to prices that equal values in $[v_1, v_2]$.

By the strict MLRP we have that, for all $v \in [v_1, v_2]$ and all $v' > v$,

$$\begin{aligned}
0 &= \int_{\mathcal{T}} F(s^a(v,t)|v)d\mu - F(\alpha(v,\eta)|v) = \\
&= \int_{\mathcal{T}} \int_{\alpha(v,\eta)}^{s^a(v,t)\vee\alpha(v,\eta)} f(x|v)dxd\mu - \int_{\mathcal{T}} \int_{\alpha(v,\eta)\wedge s^a(v,t)}^{\alpha(v,\eta)} f(x|v)dxd\mu \\
&\leq \int_{\mathcal{T}} \int_{\alpha(v,\eta)}^{s^a(v,t)\vee\alpha(v,\eta)} f(x|v)\frac{f(x|v')}{f(x|v)}dxd\mu - \int_{\mathcal{T}} \int_{\alpha(v,\eta)\wedge s^a(v,t)}^{\alpha(v,\eta)} f(x|v)\frac{f(x|v')}{f(x|v)}dxd\mu \\
&= \int_{\mathcal{T}} F(s^a(v,t)|v')d\mu - F(\alpha(v,\eta)|v').
\end{aligned}$$

Therefore, $\int_{\mathcal{T}} D_2 F(s^a(v,t)|v)d\mu \geq D_2 F(\alpha(v,\eta)|v)$, which implies $\frac{d}{dv}B^a(v|v) \geq D_2 F(\alpha(v,\eta)|v)$, a contradiction. ■

Proof of Proposition 2. The proof is divided into two cases, depending on the value of η. For $\eta \in [0, \underline{\eta}]$, where $\underline{\eta} > 0$ is given by (33), I show that there is no mispricing; whereas when $\eta > \underline{\eta}$ there is mispricing. In the latter case, I provide an algorithm to find monotone equilibrium prices satisfying *Proposition 1* and *Corollary 1* and show that they exist and are unique. After that, I show that, except for a very particular class of naïve distributions, there exists $\overline{\eta} < 1$ such that for all $\eta \geq \overline{\eta}$ sophisticated bids are confined outside the range of equilibrium prices, implying that $\mathcal{V} = [0,1]$.

Before turning into these cases, a prerequisite for existence is that any equilibrium prices satisfying *Proposition 1* and *Corollary 1* are in fact increasing. This is guaranteed if any block-monotonic distribution of sophisticated bids leads to market clearing prices that are increasing. According to *Proposition 1*, market prices in mispricing intervals are given by

$$H(p|v) = \frac{1 - \gamma - (1-\eta)F(s_k^*|v)}{\eta}. \tag{34}$$

Given any $\eta \in (0,1)$, the right-hand side of this expression is constant for $s_k^* \in \{0,1\}$ and strictly increasing in v for $s_k^* \in (0,1)$. Hence, when $H(\cdot|\cdot)$ satisfies *Assumption 4*, the resulting price is increasing in v.

Now I turn into the two cases to be considered, $\eta \in [0, \underline{\eta}]$ and $\eta \in (\underline{\eta}, 1]$.

Case 1: ($\eta \in [0, \underline{\eta}]$). By *Lemma 5*, $\alpha(\cdot, \eta)$ is non-decreasing for all $\eta \leq \underline{\eta}$. This implies that there no mispricing in any monotone equilibrium for all $\eta < \underline{\eta}$, with the distribution of sophisticated bids satisfying $B^*(v|v) = F(\alpha(v,\eta)|v)$. In the absence of mispricing and since agents cannot affect the price, no sophisticated trader has an incentive to deviate and, hence, any profile of bidding strategies yielding B^*

constitutes a BNE. One such profile is given by $\beta(s,t)=\beta(s)$ for all t, with

$$\beta(s)=\begin{cases}0 & \text{if } s\in[0,\alpha(0,\eta)]\\ v \text{ s.t. } \alpha(v,\eta)=s & \text{if } s\in(\alpha(0,\eta),\alpha(1,\eta))\\ 1 & \text{if } s\in[\alpha(1,\eta),1]\end{cases} \tag{35}$$

This takes care of existence of monotone equilibrium for $\eta\in[0,\underline{\eta}]$.

Regarding uniqueness of monotone equilibrium prices, assume there exists a monotone equilibrium with $\rho(v,\eta)\neq v$ a.e. in $(\underline{v}_1,\overline{v}_1)$ with $\underline{v}_1<\overline{v}_1$ for some $\eta\leq\underline{\eta}$. If $\rho(\underline{v}_1,\eta)<\rho(\overline{v}_1,\eta)$, by *Lemma 3*, the mass of sophisticated bids placed in $[\rho(\underline{v}_1,\eta),\rho(\overline{v}_1,\eta)]$ is zero. Since the distribution of sophisticated bids is block-monotonic (*Proposition 1*), all sophisticated traders with signals below (above) some signal s_1^* bid below $\rho(\underline{v}_1,\eta)$ (above $\rho(\overline{v}_1,\eta)$). However, given that $\alpha(\cdot,\eta)$ is increasing, we have that $s_1^*>\alpha(\underline{v}_1,\eta)$ and/or $s_1^*<\alpha(\overline{v}_1,\eta)$.[57] When $s_1^*>\alpha(\underline{v}_1,\eta)$ then $\rho(\underline{v}_1,\eta)<\underline{v}_1$ if $\underline{v}_1>0$ or $\rho(v,\eta)=0$ in $[0,v')$ for some $v'>0$ if $\underline{v}_1=0$, contradicting *Corollary 1*. On the other hand, if $s_1^*<\alpha(\overline{v}_1,\eta)$ then $\rho(\overline{v}_1,\eta)>\overline{v}_1$ if $\overline{v}_1<1$ or $\rho(v,\eta)=1$ in $(v'',1]$ for some $v''<1$ if $\overline{v}_1=1$, which again violates *Corollary 1*. Therefore, the only possibility left is that $\rho(v_1,\eta)=\rho(v_2,\eta)$, i.e. there exist an atom in the distribution of prices. But, according to *Lemma 4*, this can only happens when $\eta\geq\min\{\gamma,1-\gamma\}$, i.e. when $\eta>\underline{\eta}$.

Hence, when $\eta\in[0,\underline{\eta}]$ any monotone equilibrium price satisfies $\rho(v,\eta)=v$.

Case 2: ($\eta\in(\underline{\eta},1]$). By part (ii) of *Lemma 5* $\alpha(\cdot,\eta)$ is either non-monotonic or decreasing. Hence, there exist a non-null set of asset values that are mispriced in equilibrium, given *Lemma 6*. The following algorithm identifies the values $\{\underline{v}_k\}_{k=1}^K$, $\{\overline{v}_k\}_{k=1}^K$ and signals $\{s_k^*\}_{k=1}^K$ that satisfy the conditions of *Proposition 1* and *Corollary 1*, which characterize equilibrium prices. Then I show that these values and signals always exist and are unique. Finally, it is easy to check that the bidding strategies given by (11) implement equilibrium prices.

The steps of the algorithm are:

1. Find asset values $\{v_i^m\}_{i=1}^I$ and $\{v_i^M\}_{i=1}^{I'}$ at which $\alpha(\cdot,\eta)$ reaches a local minimum and a local maximum, respectively. If $\alpha(\cdot,\eta)$ is not well-defined in an interval (v',v'') with $\alpha(v',\eta)$ or $\alpha(v'',\eta)\in\{0,1\}$, let v' be the "unique" local maximum in $[v',v'']$ when $\alpha(v'',\eta)=1$ and v'' be the "unique" local minimum when $\alpha(v',\eta)=0$.[58] Let $v_0^m=0$ and $v_{I'+1}^M=1$.[59]

[57] Note that $\alpha(\cdot,\eta)$ is strictly increasing for $\eta<\underline{\eta}$. If $\eta=\underline{\eta}$ and $\alpha(\cdot,\underline{\eta})$ is constant in $[\underline{v}_1,\overline{v}_1]$ then $s_1^*=\alpha(\underline{v}_1,\underline{\eta})$ would involve $\rho(v,\underline{\eta})=v$ in $[\underline{v}_1,\overline{v}_1]$. Hence, one of these inequalities still needs to hold for $\rho(v,\underline{\eta})\neq v$ a.e. in $[\underline{v}_1,\overline{v}_1]$.

[58] Note that when $\alpha(v'',\eta)=1$ either $\alpha(v',\eta)=1$ or it is not well-defined. Similarly, when $\alpha(v',\eta)=0$, when $\alpha(v'',\eta)=0$ or it is not well-defined.

[59] By the continuity of $\alpha(\cdot,\eta)$, $v_i^m<v_i^M$ for all i if $\alpha(0,\eta)$ is a local minimum, and $v_i^M<v_i^m$ for all i if $\alpha(0,\eta)$ is a local maximum.

2. For each interval $\{[v_{i-j}^m, v_{i+1}^M]\}_{i=1}^{I-1+j}$, with $j = 0$ if $v_1^m = 0$ and $j = 1$ if $v_1^M = 0$, find signal values $\{s_i\}_{i=1}^{I-1+j}$ such that, when $\rho(v,\eta)$ satisfies $1-\gamma = \eta H(\rho(v,\eta)|v) + (1-\eta)F(s_i|v)$, are given by

$$s_i = \begin{cases} 0 & \textit{if } \mathbb{E}((V-\rho(V,\eta))1_{\{V\in[v_{i-j}^m, v_{i-j+1}^m]\}}|0) > 0, \\ 1 & \textit{if } \mathbb{E}((V-\rho(V,\eta))1_{\{V\in[v_i^M, v_{i+1}^M]\}}|1) < 0, \\ s & \textit{if } \mathbb{E}((V-\rho(V,\eta))1_{\{V\in[\underline{\nu}_i(s), \overline{\nu}_i(s)]\}}|s) = 0, \end{cases} \tag{36}$$

where $\underline{\nu}_i(s), \overline{\nu}_i(s)$ are respectively given by

$$\underline{\nu}_i(s) = \begin{cases} v_{i-j}^m & \textit{if } \alpha(v_{i-j}^m, \eta) > s, \\ v \in [v_{i-j}^m, v_i^M] \textit{ s.t. } \alpha(v,\eta) = s & \textit{otherwise,} \end{cases} \tag{37}$$

and

$$\overline{\nu}_i(s) = \begin{cases} v_{i+1}^M & \textit{if } \alpha(v_{i+1}^M, \eta) < s, \\ v \in [v_{i-j+1}^m, v_{i+1}^M] \textit{ s.t. } \alpha(v,\eta) = s & \textit{otherwise.} \end{cases} \tag{38}$$

3. If $s_i > s_{i+1}$ merge intervals $[v_{i-j}^m, v_{i+1}^M]$ and $[v_{i+1-j}^m, v_{i+2}^M]$ and redefine $s_i = s_i'$ and $\overline{\nu}_i(s_i') = \overline{\nu}_{i+1}(s_i')$, with s_i' given by

$$s_i' = \begin{cases} 0 & \textit{if } \mathbb{E}((V-\rho(V,\eta))1_{\{V\in[v_{i-j}^m, v_{i-j+2}^m]\}}|0) > 0, \\ 1 & \textit{if } \mathbb{E}((V-\rho(V,\eta))1_{\{V\in[v_i^M, v_{i+2}^M]\}}|1) < 0, \\ s & \textit{if } \mathbb{E}((V-\rho(V,\eta))1_{\{V\in[\underline{\nu}_i(s), \overline{\nu}_{i+1}(s)]\}}|s) = 0. \end{cases} \tag{39}$$

Repeat this step until $s_i \leq s_{i+1}$ for $i = 1, ..., K$, with K being the new number of intervals.

4. Define $s_k^* = s_k$, $\underline{v}_k = \underline{\nu}_k(s_k)$ and $\overline{v}_k = \overline{\nu}_k(s_k)$, $k = 1, ..., K$.

Several things are worth noting. First, each interval $[v_{i-j}^m, v_{i+1}^M]$ contains v_{i-j+1}^m and v_i^M. Thus, $\alpha(\cdot,\eta)$ is increasing in $(v_{i-j}^m, v_i^M) \cup (v_{i-j+1}^m, v_{i+1}^M)$ and decreasing in (v_i^M, v_{i-j+1}^m).[60] This implies $s_i \in \big[\alpha(v_{i-j+1}^m, \eta), \max\{\alpha(v_i^M,\eta), \alpha(v_{i+1}^M,\eta)\}\big]$. Assume otherwise that $0 < s_i < \alpha(v_{i-j+1}^m, \eta) < 1$. Then $\rho(v,\eta) > v$ in $[\underline{\nu}_i(s_i), \overline{\nu}_i(s_i)]$ since $\alpha(v,\eta)$ is above s_i in $[\underline{\nu}_i(s_i), v_{i+1}^M(s_i)]$. But this in turn leads to $\mathbb{E}((V - \rho(V,\eta))1_{\{V\in[\underline{\nu}_i(s_i),\overline{\nu}_i(s_i)]\}}|s_i) < 0$ when $s_i < 1$, which violates (36). Given these bounds

[60] Note that for $i = 0$, $v_{i-j}^m = v_i^M$ when $v_1^M = 0$, and $v_{i-j+1}^m = v_{i+1}^M$ for $i = 1$ when $v_K^m = 1$.

on s_i, there is a unique value $\nu_i'(s_i) \in (v_i^M, v_{i-j+1}^m)$ such that $\alpha(\nu_i'(s_i), \eta) = s_i$. Accordingly, $\rho(v, \eta) > v$ in $(\underline{\nu}_i(s_i), \nu_i'(s_i))$ and $\rho(v, \eta) < v$ in $(\nu_i'(s_i), \overline{\nu}_i(s_i))$.[61]

Second, $\underline{\nu}_i(\cdot)$ and $\overline{\nu}_i(\cdot)$ are increasing, while $v_i'(\cdot)$ is decreasing. By the continuity assumptions and *Fact 1*, each tuple $(s_i, \underline{\nu}_i(s_i), \overline{\nu}_i(s_i))$ exists and it is unique. To see why, note that as s_i grows the interval where prices are above values $(\underline{\nu}_i(s_i), \nu_i'(s_i))$ shrinks while $(\nu_i'(s_i), \overline{\nu}_i(s_i))$ grows. Furthermore, as s_i grows the probability mass (conditional on s_i) associated to $(\nu_i'(s_i), \overline{\nu}_i(s_i))$ grows relative to the mass associated to $(\underline{\nu}_i(s_i), \nu_i'(s_i))$, by the MLRP of $F(\cdot|s_i)$. Therefore, there is a unique signal s_i (which in turn uniquely determines $\underline{\nu}_i(s_i)$ and $\overline{\nu}_i(s_i)$) satisfying (36).

Third, when two adjacent intervals with signals s_i, s_{i+1} are merged (step 3 of the algorithm), the new pivotal signal s_i' lies in (s_{i+1}, s_i). Thus, any subinterval of $[\underline{\nu}_i(s_i'), \overline{\nu}_{i+1}(s_i')]$ with $\rho(v, \eta) < v$ is preceded by a subinterval with $\rho(v, \eta) > v$, which means that we can apply the same existence and uniqueness argument to the tuple $(s_i', \underline{\nu}_i(s_i'), \overline{\nu}_{i+1}(s_i'))$.

Finally, $\alpha(\cdot, \eta)$ is increasing on $[0, \underline{\nu}_1(s_1)]$, $[\overline{\nu}_i(s_i), \underline{\nu}_{i+1}(s_i)]$ and also on $[\underline{\nu}_K(s_K), 1]$. That is, it is increasing in $[0,1] \setminus \bigcup\limits_k [\underline{v}_k, \overline{v}_k]$, which implies that prices equal values in such set (*Lemma 6*).

Given all of this, (36)-(39) imply that $\{(s_k^*, \underline{v}_k, \overline{v}_k)\}$ satisfy (7)-(9). Moreover, prices given by (6) are monotonic and satisfy *Corollary 1*.

Since this algorithm provides a unique solution, we need to show that a collection $\{(s_h', \underline{v}_h', \overline{v}_h')\}$ not satisfying (36)-(39) violates (7)-(9) or *Corollary 1*.

Assume that there is a collection $\{(s_h', \underline{v}_h', \overline{v}_h')\}$ satisfying *Proposition 1*. If $s_h' \in (0,1)$ then $\mathbb{E}((V - \rho(V, \eta)) 1_{\{V \in [\underline{v}_h', \overline{v}_h']\}} | s_h') = 0$ by (9). In addition, (7)-(8) and *Corollary 1* require that $\alpha(\underline{v}_h', \eta) \geq s_h'$ with equality when $\underline{v}_h' \in (0,1)$ and $\alpha(\overline{v}_h', \eta) \leq s_h'$ with equality when $\underline{v}_h' \in (0,1)$. *Corollary 1* further requires $\alpha(\cdot, \eta)$ to be increasing at $\underline{v}_h'$ and $\overline{v}_h'$ whenever it is equal to s_h'. All these conditions imply that $\underline{v}_h \in [v_{i-j}^m, v_i^M]$ and $\overline{v}_h \in [v_{l-j}^m, v_l^M]$ for some i, l with $i < l$. But then, if $i = l + 1$, $(s_h', \underline{v}_h', \overline{v}_h') = (s_i, \underline{\nu}_i, \overline{\nu}_i)$ given (36)-(39). On the other hand, if $i < l + 1$ let s_k, $k = i, ..., l$, be the signals given by (36). If $s_k < s_{k+1}$ for all k then $s_h' \in (s_i, s_l)$ for $\mathbb{E}((V - \rho(V, \eta)) 1_{\{V \in [\underline{v}_h', \overline{v}_h']\}} | s_h') = 0$ to hold. But then $\mathbb{E}((V - \rho(V, \eta)) 1_{\{V \in [\underline{v}_h', \overline{\nu}_i(s_h')]\}} | s_h') > 0$ by *Fact 1* and, in turn, $\mathbb{E}((V - \rho(V, \eta)) 1_{\{V \in [\overline{\nu}_i(s_h'), \overline{v}_h']\}} | s_h') < 0$. Thus, a sophisticated trader receiving s_h' would rather bid $\overline{\nu}_i(s_h')$ than bid below $\underline{v}_h'$, contradicting that $\{(s_h', \underline{v}_h', \overline{v}_h')\}$ correspond to equilibrium prices. Assume then that there exists some $i \leq h \leq l$ such that $s_h \geq s_{h+1}$. In such case, abusing notation, let

[61] This is also true when $s_i \in \{0, 1\}$. Given (37)-(38), $s_i = 0$ implies that $\underline{\nu}_i = v_{i-j}^m$ and $\frac{1-\gamma-\eta H(v)}{1-\eta} < 0$ in some interval (v', v_i^m) (otherwise (36) would be violated), which leads to $\overline{\nu}_i = v_i^m$ (according to step 1 of the algorithm, v_i^m is the upper bound of the interval of values where $\alpha(\cdot, \eta)$ is not well-defined). The latter implies that $\rho(v, \eta) < v$ in $(v', \overline{\nu}_i)$. Since $\alpha(\cdot, \eta)$ is either increasing in (v_{i-j}^m, v_i^M) or above 0 when $v_{i-j}^m = 0$ (part (i) of *Lemma 5*), $\alpha(v, \eta) > 0$ (and thus $\rho(v, \eta) > v$) in $(\underline{\nu}_i, v')$. Similarly, $s_i = 1$ implies that $\underline{\nu}_i = v_i^M$ and $\frac{1-\gamma-\eta H(v)}{1-\eta} > 1$ in some interval (v_i^M, v'), which means that $\overline{\nu}_i = v_{i+1}^M$. Hence, $\rho(v, \eta) > v$ in (v_i^M, v') and $\rho(v, \eta) < v$ in (v', v_{i+1}^M).

$\{s_{i'}\}$ denote the new collection of signals, associated to intervals $[v^m_{i-j}, v^M_{i+1}]$ included in $[\underline{v}'_h, \overline{v}'_h]$, given by (39) after merging intervals $[v^m_{h-j}, v^M_{h+1}]$ and $[v^m_{h+1-j}, v^M_{h+2}]$. If $s_{i'} < s_{i'+1}$ for some i' in the new collection of signals, we again have that $\mathbb{E}((V - \rho(V,\eta))1_{\{V \in [\overline{\nu}_{i'}(s'_h), \overline{v}'_h\}}]|s'_h) < 0$, which leads to a profitable deviation by a trader receiving signal s'_h. By using this argument iteratively, we arrive at the conclusion that $(s_i, \underline{\nu}_i(s_i), \overline{\nu}_i(s_i))$ is the unique tuple satisfying (37)-(39), which are equivalent to (7)-(9), and that is compatible with the equilibrium behavior of sophisticated traders. Hence, $\{(s'_h, \underline{v}'_h, \overline{v}'_h)\}$ cannot part of a characterization of equilibrium prices if $(s'_h, \underline{v}'_h, \overline{v}'_h) \neq (s_i, \underline{\nu}_i(s_i), \overline{\nu}_i(s_i))$.

If $s'_h = 0$ then $\underline{v}'_h = 0$ by part (i) of *Lemma 5*. We also have that $\mathbb{E}((V - \rho(V,\eta))1_{\{V \in [0, \overline{v}'_h\}}]|0) \geq 0$ by (9). In addition, *Corollary 1* requires that $\rho(v,\eta) < v$ in the upper part of $[0, \overline{v}'_h)$, which means that $\overline{v}'_h \in [v^m_{i-j}, v^m_{i-j+1}]$ for some $i = 1, ..., I-1+j$. But this can only happen if $\frac{1-\gamma-\eta H(v)}{1-\eta} < 0$ in some interval (v', v^m_{i-j+1}). Thus, $\overline{v}'_h = v^m_{i-j+1}$, otherwise $\overline{v}'_1$ would not satisfy (8). We need to consider two cases. If $i = 1$ we have that the unique triplet satisfying these conditions is $(s_1, \underline{v}_1, \overline{v}_1)$ as defined by the above algorithm. If $i > 1$ and $s_l < s_{l+1}$ for all $l < i$, then $\mathbb{E}((V - \rho(V,\eta))1_{\{V \in [\underline{\nu}_{l+1}(s_{l+1}), \overline{\nu}_{l+1}(s_{l+1})]\}}|s_{l+1}) = 0$ with $\underline{\nu}_{l+1}(s_{l+1}) < v^m_{i-j+1}$ (otherwise $\rho(v,\eta) < v$ a.e. in $[\underline{\nu}_{l+1}(s_{l+1}), \overline{\nu}_{l+1}(s_{l+1})]$) and a trader receiving a signal in (s_l, s_{l+1}) would rather deviate and bid $\underline{\nu}_i(s_{l+1})$. Therefore, $s_l > s_{l+1} = 0$ for some $l < i$. Using iterative merging we arrive at the conclusion that either $(s'_h, \underline{v}'_h, \overline{v}'_h) = (s^*_1, \underline{v}_1, \overline{v}_1)$ or that $(s'_h, \underline{v}'_h, \overline{v}'_h)$ violates *Proposition 1*.

Finally, when $s'_h = 1$ we have that $\overline{v}'_h = 1$ by part (i) of *Lemma 5* and $\mathbb{E}((V - \rho(V,\eta))1_{\{V \in [\underline{v}'_h\}}]|1) \leq 0$ by (9). The latter implies that $\frac{1-\gamma-\eta H(v)}{1-\eta} > 1$ in an interval $(v^M_{i-j'}, v')$ for some $i = 1, ..., I'$ with $j' = 0$ if $v^M_{I'} < 1$ and $j' = 1$ otherwise, whereas *Corollary 1* requires that $\rho(v,\eta) > v$ in the lower part of $(\underline{v}'_h, 1]$. Hence, $\underline{v}'_h = v^M_i$ by (7). When $i = I'$, $(s_K, \underline{v}_K, \overline{v}_K)$ is the only triplet satisfying the above conditions. If $i < I'$ it has to be that $s_{l+1} < s_l = 1$ for some $l \geq i$, otherwise a trader receiving a signal in (s_l, s_{l+1}) would rather deviate and bid $\overline{\nu}_{l+1}(s_{l+1}) > v^M_i$, given *Fact 1* and that $\mathbb{E}((V - \rho(V,\eta))1_{\{V \in [\underline{\nu}_{l+1}(s_l), \overline{\nu}_l(s_l)]\}}|s_l) = 0$. Using the above merging argument it has to be that either $(s'_h, \underline{v}'_h, \overline{v}'_h) = (s^*_K, \underline{v}_K, \overline{v}_K)$ or that it violates *Proposition 1*.

This completes the proof that a collection $\{(s^*_k, \underline{v}_k, \overline{v}_k)\}$ satisfying (7)-(9) exists and it is unique. An example of bidding strategies yielding such equilibrium prices is given by (11).

To complete the proof of *Proposition 2*, we need to show that there exists $\overline{\eta}$ such that $\mathcal{V} = [0,1]$ for all $\eta \geq \overline{\eta}$. By *Lemma 6*, prices that equal values can only exist for values such that $\alpha(\cdot,\eta)$ is increasing. In addition, by *Lemma 5*, once $\alpha(\cdot,\eta)$ is decreasing at v it is decreasing for all $\eta' > \eta$. Therefore, if there exists a share $\overline{\eta}$ such that $\alpha(\cdot,\eta)$ is either decreasing or not well-defined, it will also be so for all $\eta > \overline{\eta}$. In this context, the mass of bids at $[0, \rho(0,\eta)]$ (resp. $[\rho(1,\eta), 1]$) is given by the mass of signals $s \leq s_1^*$ ($s > s_1^*$), where

$$s_1^* = \begin{cases} 0 & \textit{if } \mathbb{E}(V - \rho(V,\eta)|s) > 0 \;\; \forall s, \\ 1 & \textit{if } \mathbb{E}(V - \rho(V,\eta)|s) < 0 \;\; \forall s, \\ s \;\textit{s.t.}\; \mathbb{E}(V - \rho(V,\eta)|s) = 0 & \textit{otherwise,} \end{cases} \tag{40}$$

with $\rho(v,\eta)$ satisfying $1-\gamma = \eta H(\rho(v,\eta)|v) + (1-\eta)F(s_i|v)$. The signal s_1^* exists and it is unique as shown above.

Note that, by *Lemma 5*, if for any v such that $H(v) = 1-\gamma$ we have that $H'(v) > 0$, then there exists $\overline{\eta} < 1$ such that $\alpha(\cdot,\eta)$ is decreasing for all $\eta > \underline{\eta}$, leading to complete mispricing (*Lemma 6*).

If, however, $H'(v) < 0$ for at least a value v satisfying $H(v) = 1-\gamma$, a region without mispricing may exist around v for all $\eta < 1$. To see why, note that for any such value there are two values v' and v'' with $v' < v < v''$ such that $H(v') = H(v'') = 1-\gamma$ and $H(v'), H(v'') > 0$. Accordingly, for η close to one, $\alpha(\cdot,\eta)$ is decreasing in a neighborhood of v' and v'' and increasing in a neighborhood of v, and by its continuity, its range in these neighborhoods is the whole unit interval. Thus there are at least two intervals $\{[v_{i-j}^m, v_{i+1}^M]\}_{i=1}^{I-1+j}$, $i = 1,2$ as defined in the above algorithm. If the two signals satisfying (36) for each interval are such that $s_1 < s_2$, there exists a region with no mispricing in the interval $[\overline{\nu}(s_1), \underline{\nu}(s_2)]$ with $\underline{\nu}(s_2)$, $\overline{\nu}(s_1)$ given by (37) and (38), respectively.

Hence, if $H'(v) > 0$ for any value v such that $H(v) = 1-\gamma$, then $\overline{\eta} < 1$ whereas $\overline{\eta}$ might be equal to one otherwise. ■

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