

Sorting by Search Intensity*

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Abstract

In this paper, I analyze a general equilibrium on-the-job search model with endogenous search intensity and heterogeneous workers and firms. Wages are set through bargaining as in Cahuc, Postel-Vinay, and Robin (2006). I provide proof of existence and uniqueness of steady state equilibrium. Given equally efficient search on and off the job, I provide proof that the equilibrium firm productivity distribution is stochastically increasing (decreasing) in worker skill if the match production function is supermodular (submodular). By implication, if the production function is supermodular (submodular) high skill workers on average match with more (less) productive firms. I also show that this strong notion of sorting does not obtain everywhere for the reverse conditioning.

Keywords: Assortative matching, supermodularity, search friction, endogenous search intensity.

JEL Classification Numbers: C78, D51, D83, J31, J62, J63, J64.

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1 Introduction

In this paper, I present a general equilibrium on-the-job search model with worker skill and firm productivity heterogeneity. I show that if the match production function is supermodular then positive sorting obtains in the sense that higher skilled workers match with more productive firms. If the production function is submodular, negative sorting obtains.¹ Specifically, the result is stated in terms of stochastic dominance; if the production function is supermodular (submodular) then the worker skill conditional equilibrium firm productivity distribution is stochastically increasing (decreasing) in worker skill. If the production function is modular, then no sorting results. Interestingly, this strong notion of sorting does not always obtain everywhere for the reverse conditioning.

Workers differ in their permanent skill level, h . Firms differ in their productivity level p . A match between a worker and firm produces output $f(h, p)$. Wages are in the model set through bargaining where outside offers can alter the worker's threat point and trigger renegotiation. Firms are assumed to face a constant returns to scale production technology and will match with any worker as long as match surplus is positive. Workers can affect the arrival rate of meetings with employers through a costly choice of search intensity. The model allows that search can be more or less efficient when unemployed relative to search on the job. If search is more efficient in the unemployed state, the model implies a non-trivial reservation productivity level choice.

If the production function is supermodular, high skill workers have greater gains to matching with more productive firms and therefore search with greater intensity for outside opportunities for any given step on the firm productivity ladder. Consequently high skill workers climb higher on the productivity ladder than low skill workers in a stochastic dominance sense. In the submodular case, the low skill worker searches with greater intensity and climbs higher on the ladder. If the production function is modular, then firm productivity conditional search intensity is the same across worker skill levels, and no sorting obtains. To my knowledge this is the first paper to demonstrate the match allocation implications of this type of model.²

The sorting mechanism in this paper is a novel alternative to the partnership formation environment studied in for example Shimer and Smith (2000). It differs in two major ways: First,

¹Assuming that the production function $f(h, p)$ is smooth in worker skill h and firm productivity p , the production function is supermodular iff $f_{hp} > 0$, it is submodular iff $f_{hp} < 0$, and modular iff $f_{hp} = 0$.

²In their conclusion Cahuc, Postel-Vinay, and Robin (2006) conjecture a mechanism similar to the driving mechanism in this paper, "... if good-quality workers receive alternative offers more often, then they will climb the wage and productivity ladder faster, and positive sorting results. Solving such an equilibrium search model with sorting and estimating it is surely very difficult, but nevertheless constitutes a very promising area for future research." Of course, a key issue is to show under which circumstances good quality workers receive alternative offers more often.

sorting is a result of differential search intensities rather than matching set differences. Second, the model is fundamentally asymmetric in that sorting is driven by worker behavior, only. The partnership formation in Shimer and Smith (2000) assumes a match formation setting where both sides of the market face the same basic match formation constraints: Agents can only match with one agent at a time and can only search for partnerships when not currently in a match. While certainly a reasonable representation of for example the marriage market, the underlying assumption of extreme match opportunity scarcity is not satisfied for commonly used models of the firm. An obvious example is a firm with a constant returns to scale production technology and a costly labor adjustment process that is at least in part tied to labor market friction. An example that does not require constant returns to scale is where the firm’s production technology is such that production depends only on an additive aggregate of worker efficiency units, $F(h^n, p) = f(\sum_{i=1}^n h_i, p)$ and a friction process is maintaining steady state equilibrium productivity differences across firms on the margin. The latter example highlights that the degree of substitutability of labor in the production process is an important determinant of match opportunity scarcity at the firm level. It is of course possible to construct a multi-worker firm environment where the firm faces some degree of match opportunity scarcity and consequently will adopt a discriminating hiring policy. This type of environment would fall somewhere between the traditional partnership model and the model studied in this paper.

The paper is organized as follows: Section 2 presents the model and provides proof of existence and uniqueness of steady state equilibrium. Section 3 then investigates the sorting properties of the equilibrium. Two numerical examples are presented in Section 4, and Section 5 concludes.

2 Model

The framework of the model is an endogenous search intensity model with type heterogeneity on both the worker and firm side. The paper adopts a wage determination mechanism similar to that of Cahuc, Postel-Vinay, and Robin (2006). The model is set in continuous time.

There is a unit mass of workers. Workers and firms form matches in a frictional labor market. A worker is characterized by his or her permanent skill level h which is independently and identically distributed across workers according to the cumulative distribution function $\Psi(\cdot)$ with the normalized support, $[0, 1]$. Workers are infinitely lived and discount the future at rate r . There is mass m of firms that differ in productivity p . Firm productivity is distributed across firms according to the CDF $\Phi(\cdot)$ over the normalized support $[0, 1]$. Firms face a constant returns to scale production

technology and the decision to accept a match with a worker does not affect the profitability of future vacancies nor the ability to create them. Hence, a firm will match with any worker as long as the match produces weakly positive profits. It is assumed that each firm posts a single vacancy at any instant. This provides a particularly simple mapping between the exogenously given firm heterogeneity distribution and the vacancy offer distribution, specifically $\Gamma(p) = \Phi(p)$, $\forall p \in [0, 1]$, where $\Gamma(\cdot)$ is the vacancy offer CDF.³

To simplify the analysis assume that $\Psi(\cdot)$ and $\Phi(\cdot)$ are everywhere differentiable with probability density functions given by $\psi(h)$ and $\phi(p)$, respectively.

A match between a skill level h worker and a productivity p firm produces output $f(h, p)$. The production function is assumed to satisfy the regularity conditions in Assumption 1.

Assumption 1 *The production function is a mapping $f : [0, 1]^2 \rightarrow R_+$. f is at least twice differentiable with $f_h(h, p) > 0$ and $f_p(h, p) > 0$ for any $(h, p) \in [0, 1]^2$.*

Workers can be either employed or unemployed. Any match faces an exogenous destruction rate δ . Regardless of employment state, a worker can search for a new job. The search technology may differ across the two employment states. Specifically, a search intensity s results in the arrival rate of new job opportunities of $\kappa\lambda s$ or λs if unemployed or employed, respectively, where $\kappa > 0$. If $\kappa > 1$ then search is more efficient in the unemployed state. $\lambda > 0$ is the per unit of search intensity arrival rate of offers. Implicit in the statement that λ is constant, I am making the simplifying assumption that there are no negative crowding effects in the matching technology. The cost of a search intensity s is given by $c(s)$, where $c(\cdot)$ satisfies the conditions in Assumption 2.

Assumption 2 *The search cost function is a mapping $c : R_+ \rightarrow R_+$. The function is increasing, strictly convex, and three times differentiable with $c(0) = c'(0) = 0$.*

Employment contracts between workers and employers are set through a Rubinstein (1982) style bargaining game following the same protocol as in Cahuc, Postel-Vinay, and Robin (2006). An alternative but related bargaining protocol is presented in Yamaguchi (2006). In both cases, it is assumed that the worker can use a contact with one employer as a threat point in a bargaining game with another. An employment contract can only be re-negotiated by mutual consent. If the worker is unemployed, then the value of unemployment will be the worker's threat point. The detailed bargaining argument is presented in the appendix.

³Bagger and Lentz (2007) analyze the more general model with endogenous vacancy creation. In the context of an endogenous vacancy creation model, the case of a constant vacancy creation rate across firms is associated with a vacancy creation cost function with strong curvature, such as a function $c(v) = v^\chi$ where χ is large.

An employment contract consists of a worker's wage level and search intensity. Specifically, this implies the assumption that search intensities can be contracted upon. In the current setting it implements the between the worker and firm jointly efficient search intensity level. In the alternative case where search intensity is chosen by the worker in response to some match surplus split, the search intensity will be inefficiently high when the worker does not receive the full match surplus. As such, the setup in this paper represents the jointly efficient search intensity benchmark.

Denote by $\tilde{V}(h, p, w, s)$ a skill h worker's asset value of a job with a productivity p firm and employment contract (w, s) . The outcome of the employment contract bargaining as described in the appendix is such that the agreed upon search intensity maximizes the joint surplus of the match and the wage then dictates the surplus split. Hence, the search intensity depends only on the (h, p) pair,

$$s(h, p) = \arg \max_{s \geq 0} \tilde{V}(h, p, f(h, p), s). \quad (1)$$

If the worker is unemployed, the outside option in the bargaining is the value of unemployment. Denote by $(w_0(h, p), s(h, p))$ the employment contract of a skill h worker who was hired out of unemployment by a productivity p firm. It satisfies,

$$\tilde{V}(h, p, w_0(h, p), s(h, p)) = \beta \tilde{V}(h, p, f(h, p), s(h, p)) + (1 - \beta)V_0(h), \quad (2)$$

where $V_0(h)$ is the asset value of unemployment for a skill h worker. β is the probability that the worker gets to make an offer in the alternating offers bargaining game.

If an employed worker receives an outside offer, the worker will go to the most productive firm. The bargaining outcome is identical to that of axiomatic Nash bargaining where the worker's threat point is full surplus extraction with the less productive firm, the firm has a zero threat point, and β is the bargaining power of the worker. Denote by p and q the productivity levels of the two firms, where $p \geq q$. If the two firms are of equal productivity, the worker stays with the current firm. Denote the resulting wage by $w(h, q, p)$. It satisfies,

$$\tilde{V}(h, p, w(h, q, p), s(h, p)) = \beta \tilde{V}(h, p, f(h, p), s(h, p)) + (1 - \beta)\tilde{V}(h, q, f(h, q), s(h, q)). \quad (3)$$

In the analysis to follow it is useful to adopt the shorthand, $V(h, q, p) = \tilde{V}(h, p, w(h, q, p), s(h, p))$.

It is assumed that an unemployed skill h worker receives an income stream $f(h, 0)$. Hence aside from the potential search technology differences, unemployment is as if the worker is receiving full surplus with the lowest type firm. The unemployed worker chooses search intensity, $s_0(h)$, to maximize unemployment surplus. The Bellman equation for the value of unemployment is given

by,

$$\begin{aligned} rV_0(h) &= \max_{s \geq 0} \left\{ f(h, 0) - c(s) + \kappa \lambda s E \left[\max \{ 0, \tilde{V}(h, p, w_0(h, p), s(h, p)) - V_0(h) \} \right] \right\} \\ &= \max_{s \geq 0} \left\{ f(h, 0) - c(s) + \beta \kappa \lambda s \int_{R(h)}^1 [V(h, p', p') - V_0(h)] d\Gamma(p') \right\}, \end{aligned} \quad (4)$$

where r is the interest rate, $\Gamma(p)$ is the cumulative vacancy productivity distribution, and $R(h)$ is the skill h reservation productivity level defined by,

$$V(h, R(h), R(h)) = V_0(h). \quad (5)$$

It is straightforward to prove that $V(h, p, p)$ is monotonically increasing in p which establishes the reservation property of the model; that a skill h worker will accept a match with any employer above the productivity threshold level, $R(h)$.

Applying integration by parts and the envelope theorem, equation (4) can be restated as,

$$rV_0(h) = \max_{s \geq 0} \left\{ f(h, 0) - c(s) + \beta \kappa \lambda s \int_{R(h)}^1 \frac{f_p(h, p')[1 - \Gamma(p')] dp'}{r + \delta + \beta \lambda s(h, p')[1 - \Gamma(p')]} \right\}. \quad (6)$$

The value of employment with a productivity p firm at wage $w(h, q, p)$ and search intensity $s(h, p)$ is given by,

$$\begin{aligned} rV(h, q, p) &= w(h, q, p) - c(s(h, p)) + \lambda s(h, p) \left[\int_p^1 [V(h, p, p') - V(h, q, p)] d\Gamma(p') \right. \\ &\quad \left. + \int_q^p [V(h, p', p) - V(h, q, p)] d\Gamma(p') \right] + \delta [V_0(h) - V(h, q, p)], \end{aligned} \quad (7)$$

which by use of integration by parts and the envelope theorem can be re-written as,

$$\begin{aligned} (r + \delta)V(h, q, p) &= w(h, q, p) - c(s(h, p)) + \delta V_0(h) \\ &\quad + \lambda s(h, p) \left[\beta \int_p^1 \frac{f_p(h, p')[1 - \Gamma(p')] dp'}{r + \delta + \beta \lambda s(h, p')[1 - \Gamma(p')]} \right. \\ &\quad \left. + (1 - \beta) \int_q^p \frac{f_p(h, p')[1 - \Gamma(p')] dp'}{r + \delta + \beta \lambda s(h, p')[1 - \Gamma(p')]} \right]. \end{aligned} \quad (8)$$

The detailed derivation of equation (8) can be found in the appendix.

2.1 The search choices

The employment state conditional search intensity is found by use of equations (1) and (4). Together with equation (8), they imply the first order conditions,

$$c'(s_0(h)) = \beta \kappa \lambda \int_{R(h)}^1 \frac{f_p(h, p')[1 - \Gamma(p')] dp'}{r + \delta + \beta \lambda s(h, p')[1 - \Gamma(p')]} \quad (9)$$

$$c'(s(h, p)) = \beta \lambda \int_p^1 \frac{f_p(h, p')[1 - \Gamma(p')] dp'}{r + \delta + \beta \lambda s(h, p')[1 - \Gamma(p')]} \quad (10)$$

By assumptions 1 and 2 and by continuity of $\Gamma(\cdot)$, $s(h, p)$ is continuous in h and p . By convexity of $c(\cdot)$, differentiation of equation (10) with respect to p immediately yields that $s(h, p)$ is monotonically decreasing in p , $\forall h$. Furthermore, $s(h, 1) = 0$, $\forall h$. Lemma 1 establishes that the search intensity is strictly increasing (decreasing) in worker skill h if the production function is strictly supermodular (submodular). If the production function is modular, then the search intensity is identical across worker skill levels.

Lemma 1 *For any pair $(h_0, h_1) \in [0, 1]^2$ such that $h_0 < h_1$, and for all $p \in [0, 1]$,*

- $f_{hp}(h, p) > 0 \forall (h, p) \Rightarrow s(h_0, p) < s(h_1, p)$ (supermodular).
- $f_{hp}(h, p) < 0 \forall (h, p) \Rightarrow s(h_0, p) > s(h_1, p)$ (submodular).
- $f_{hp}(h, p) = 0 \forall (h, p) \Rightarrow s(h_0, p) = s(h_1, p)$ (modular).

For any $h \in [0, 1]$, $s(h, 1) = 0$.

Proof. Let the production function be strictly supermodular. Consider any $h_0 < h_1$. Assume contrary to the lemma that for some $p \in [0, 1]$, $s(h_0, p) \geq s(h_1, p)$. By continuity of $s(h, p)$ in p and $s(h, 1) = 0$ for any h there exists some $\hat{p} \in [p, 1]$ such that $s(h_0, \hat{p}) = s(h_1, \hat{p})$ and $s(h_0, p') \geq s(h_1, p')$ for all $p' \in [p, \hat{p}]$. By the definition of \hat{p} and by equation (10) it follows that,

$$\int_{\hat{p}}^1 \frac{f_p(h_0, p')[1 - \Gamma(p')] dp'}{r + \delta + \beta \lambda s(h_0, p')[1 - \Gamma(p')]} = \int_{\hat{p}}^1 \frac{f_p(h_1, p')[1 - \Gamma(p')] dp'}{r + \delta + \beta \lambda s(h_1, p')[1 - \Gamma(p')]}.$$

With this, one can make the argument,

$$\begin{aligned} c'(s(h_0, p)) &= \beta \lambda \left[\int_p^{\hat{p}} \frac{f_p(h_0, p')[1 - \Gamma(p')] dp'}{r + \delta + \beta \lambda s(h_0, p')[1 - \Gamma(p')]} + \int_{\hat{p}}^1 \frac{f_p(h_0, p')[1 - \Gamma(p')] dp'}{r + \delta + \beta \lambda s(h_0, p')[1 - \Gamma(p')]} \right] \\ &= \beta \lambda \left[\int_p^{\hat{p}} \frac{f_p(h_0, p')[1 - \Gamma(p')] dp'}{r + \delta + \beta \lambda s(h_0, p')[1 - \Gamma(p')]} + \int_{\hat{p}}^1 \frac{f_p(h_1, p')[1 - \Gamma(p')] dp'}{r + \delta + \beta \lambda s(h_1, p')[1 - \Gamma(p')]} \right] \\ &< \beta \lambda \left[\int_p^{\hat{p}} \frac{f_p(h_1, p')[1 - \Gamma(p')] dp'}{r + \delta + \beta \lambda s(h_1, p')[1 - \Gamma(p')]} + \int_{\hat{p}}^1 \frac{f_p(h_1, p')[1 - \Gamma(p')] dp'}{r + \delta + \beta \lambda s(h_1, p')[1 - \Gamma(p')]} \right] \\ &= c'(s(h_1, p)), \end{aligned}$$

where the inequality follows from strict supermodularity of $f(h, p)$ and that $s(h_0, p') \geq s(h_1, p')$, $\forall p' \in [p, \hat{p}]$. By strict convexity of $c(\cdot)$ this contradicts the assumption of $s(h_0, p) \geq s(h_1, p)$. Hence, it has been shown by contradiction that for any $h_0 < h_1$ it must be that $s(h_0, p) < s(h_1, p)$, $\forall p \in [0, 1]$.

The submodular case can be established with an analogous proof. The proof of the claim for the modular case follows trivially from inspection of equation (10). ■

Lemma 2 characterizes the reservation productivity level $R(h)$ defined in equation (5).

Lemma 2 For any $h \in [0, 1]$, if $\kappa \leq 1$ then $R(h) = 0$, and if $\kappa > 1$ then $1 > R(h) > 0$. Furthermore, if for any pair $(h_0, h_1) \in [0, 1]^2$ and for all $p \in [0, 1]$ $f_p(h_0, p) = f_p(h_1, p)$, then $R(h_0) = R(h_1)$. The last statement says that in the modular case, the productivity threshold is the same across worker skill levels.

Proof. The proof follows trivially from re-writing equation (5) as,

$$\int_0^{R(h)} f_p(h, p') dp' = \max_{s \geq 0} \left\{ -c(s) + \beta \kappa \lambda s \int_{R(h)}^1 \frac{f_p(h, p') [1 - \Gamma(p')] dp'}{r + \delta + \beta \lambda s (h, p') [1 - \Gamma(p')] } \right\} - \max_{s \geq 0} \left\{ -c(s) + \beta \lambda s \int_{R(h)}^1 \frac{f_p(h, p') [1 - \Gamma(p')] dp'}{r + \delta + \beta \lambda s (h, p') [1 - \Gamma(p')] } \right\}. \quad (11)$$

$R(h) = 1$ implies that the right hand side of equation (11) is zero while the left hand side is strictly positive. Hence $R(h) = 1$ cannot be a solution to equation (11) for any h . If $\kappa \leq 1$ then the right hand side is zero or negative for any h and since that the production function is strictly increasing in p it must be that $R(h) = 0$. If $\kappa > 1$ the right hand side is strictly positive for any $R(h) \in [0, 1]$. Hence, the left hand side must be strictly positive in the solution as well, implying $R(h) > 0$ for any h .

In the case where $f_p(h_0, p) = f_p(h_1, p)$ for all p , it follows from Lemma 1 that $s(h_0, p) = s(h_1, p)$ for all p and consequently equation (11) is identical for h_0 and h_1 implying $R(h_0) = R(h_1)$. Therefore, in the modular case, the reservation productivity level is identical across worker skill levels. ■

In the case where $\kappa > 1$, an obvious question of interest is how $R(h)$ varies with h . Lemma 2 states that in the absence of production function complementarities, $R(h)$ is identical across worker skill levels. The model in this paper shares an important feature with that of Shimer and Smith (2000) in that rejecting an offer is made costly by the passage of time and discounting. The reservation productivity level is found at the point where the marginal benefit of rejecting an offer is exactly offset by the cost of waiting for the next acceptable offer. As also pointed out in Atakan (2006), the value of time in this kind of model varies across workers, and consequently any type of positive complementarity may not be sufficient to produce an increasing relationship between $R(h)$ and h . It is indeed quite easy to produce counterexamples in the model where supermodularity of the production function is not enough to produce an everywhere weakly increasing relationship between $R(h)$ and h .⁴

⁴One such example is $f(h, p) = (h + p)^2$ combined with an appropriately chosen specification of the remainder of the model.

2.2 Steady state

Denote by $G(h, q, p)$ the joint CDF of matches between skill level h workers and productivity p firms where q is the productivity of the second most productive firm the worker has met during the current employment spell. Denote by $g(h, q, p)$ the corresponding pdf. By definition, $q \leq p$. In the absence of a second most productive firm, $q = R(h)$. Denote by $v(h)$ the pdf of the skill distribution of unemployed workers and by $\Upsilon(h) = \int_0^h v(h')dh'$ the CDF. Finally, let the unemployment rate be given by u . In steady state, the flow into the mass $G(h, q, p)$ must equal the outflow. Hence, the steady state condition on the joint CDF of matches is,

$$(1-u)\delta G(h, q, p) + (1-u)\lambda \int_0^h \int_{R(h')}^q \left\{ [1-\Gamma(p)] \int_{q'}^p s(h', p') g(h', q', p') dp' + [1-\Gamma(q)] \int_q^p s(h', p') g(h', q', p') dp' \right\} dq' dh' = u\kappa\lambda \int_0^h \mathbb{I}[R(h) \leq p] s_0(h') [\Gamma(p) - \Gamma(R(h'))] d\Upsilon(h'), \quad (12)$$

where the left hand side represents the outflow and the right hand side the inflow. The indicator function $\mathbb{I}[\cdot]$ equals one if its expression is true, zero otherwise. The outflow consists of exogenous separation and workers either quitting to go to firms of productivity greater than p , or if worker meets an outside firm with productivity greater than q given current employment with a greater than q productivity firm. The inflow consists of unemployed skill h or less workers with reservation productivity levels less or equal to q that meet productivity p or less vacancies.

Equation (12) implies that steady state unemployment satisfies,

$$(1-u)\delta = u\kappa\lambda \int_0^1 s_0(h) [1-\Gamma(R(h))] d\Upsilon(h). \quad (13)$$

Using equation (13), one can re-write equation (12) as,

$$\int_0^h \int_{R(h')}^q \left\{ \int_{q'}^p [\delta + \lambda s(h', p') [1-\Gamma(p)]] g(h', q', p') dp' + \int_q^p [\delta + \lambda s(h', p') [1-\Gamma(q)]] g(h', q', p') dp' \right\} dq' dh' = \frac{\delta \int_0^h \mathbb{I}[R(h) \leq p] s_0(h') [\Gamma(p) - \Gamma(R(h'))] d\Upsilon(h')}{\int_0^1 s_0(h) [1-\Gamma(R(h))] d\Upsilon(h)}. \quad (14)$$

It is worth exploring equation (14) for a few special cases. In particular, consider the case where the search technology is the same across employment states, $\kappa = 1$. This implies that $R(h) = 0, \forall h$.

In this case, equation (14) simplifies to,

$$\int_0^h \int_b^q \left\{ \int_{q'}^q [\delta + \lambda s(h', p') [1 - \Gamma(p)]] g(h', q', p') dp' + \int_q^p [\delta + \lambda s(h', p') [1 - \Gamma(q)]] g(h', q', p') dp' \right\} dq' dh' = \delta \Gamma(p) \frac{\int_0^h s_0(h') d\Upsilon(h')}{\int_0^1 s_0(h) d\Upsilon(h)}. \quad (15)$$

In addition to the assumption of $\kappa = 1$, consider the modular production function case where $s_0(h) = s_0$ and $s(h, p) = s(p)$ for all h . In this case, one immediately obtains that $G(h, 1, 1) = \Upsilon(h)$, that is, the worker skill distribution is identical across employment states. This follows directly from $s_0(h) = s_0$. Furthermore, there is no assortative matching in that the skill distribution is independent of firm types, $G(h, q, p) = \Upsilon(h)G(q, p)$. In this case, equation (14) gives the following expression for $G(q, p)$,

$$G(q, p) = \Gamma(p) - \frac{\lambda}{\delta} \int_0^q \left[[1 - \Gamma(p)] \int_{q'}^q s(p') g(q', p') dp' + [1 - \Gamma(q)] \int_q^p s(p') g(q', p') dp' \right] dq'.$$

In particular, the distribution of matches with productivity p or less is,

$$G(p, p) = \Gamma(p) - \frac{\lambda}{\delta} [1 - \Gamma(p)] \int_0^p \int_{q'}^p s(p') g(q', p') dp' dq'.$$

If in addition to the already stated assumptions, one adds the assumption of exogenous search intensity $s(p') = s$, one obtains

$$G(p, p) = \frac{\delta \Gamma(p)}{\delta + \lambda s (1 - \Gamma(p))},$$

which corresponds exactly to the case in Postel-Vinay and Robin (2002).

2.3 Steady state equilibrium

In equilibrium, the overall worker skill distribution is related to the employment state conditional skill distributions by,

$$\Psi(h) = (1 - u)G(h, 1, 1) + u\Upsilon(h), \quad (16)$$

which by equation (12) implies,

$$\Upsilon(h) = \frac{\int_0^h [1 + \frac{\lambda}{\delta} s_0(h') [1 - \Gamma(R(h'))]]^{-1} d\Psi(h')}{\int_0^1 [1 + \frac{\lambda}{\delta} s_0(h') [1 - \Gamma(R(h'))]]^{-1} d\Psi(h')}. \quad (17)$$

A steady state equilibrium is defined in Definition 1.

Definition 1 *A steady state equilibrium is a tuple $\{G(h, q, p), \Upsilon(h), u, s(h, p), s_0(h), R(h), w(h, q, p)\}$ that satisfies equations (3), (5), (9), (10), (13), (14), and (17).*

Existence of a steady state equilibrium is established in Proposition 1

Proposition 1 *A unique steady state equilibrium exists.*

Proof. The search intensity $s(h, p)$ is for any $h \in [0, 1]$ a solution to the first order initial value non-linear differential equation,

$$\begin{aligned} s_p(h, p) &= \frac{-f_p(h, p)[1 - \Gamma(p)]}{c''(s(h, p))[r + \delta + \beta\lambda s(h, p)[1 - \Gamma(p)]} \\ &= S(h, p, s(h, p)), \end{aligned} \quad (18)$$

where

$$s(h, 1) = 0.$$

By Assumptions 1 and 2 and by continuity of $\Gamma(p)$, S and $\partial S/\partial s$ are continuous in p and s . Therefore, a unique solution $s(h, p)$ to equation (18) exists for any $h \in [0, 1]$.⁵

The reservation productivity level $R(h)$ is the solution to

$$V_0(h|R(h)) = V(h, R(h), R(h)), \quad (19)$$

where the notation has been slightly altered to emphasize that V_0 is conditional on the reservation productivity level, and

$$rV_0(h|R(h)) = f(h, 0) + \max_{s \geq 0} \left\{ -c(s) + \beta\kappa\lambda s \int_{R(h)}^1 \frac{f_p(h, p')[1 - \Gamma(p')] dp'}{r + \delta + \beta\lambda s(h, p')[1 - \Gamma(p')]} \right\} \quad (20)$$

$$rV(h, p, p) = f(h, p) + \max_{s \geq 0} \left\{ -c(s) + \beta\lambda s \int_p^1 \frac{f_p(h, p')[1 - \Gamma(p')] dp'}{r + \delta + \lambda\beta s(h, p')[1 - \Gamma(p')]} \right\}. \quad (21)$$

By the existence and uniqueness of $s(h, p)$ and by Assumption 2 there exists a unique search intensity solution s_0 for any choice of $R(h)$ defined implicitly by,

$$c'(s_0(h|R(h))) = \beta\kappa\lambda \int_{R(h)}^1 \frac{f_p(h, p')[1 - \Gamma(p')] dp'}{r + \delta + \beta\lambda s_0(h, p')[1 - \Gamma(p')]}.$$

By the envelope theorem, one obtains,

$$\begin{aligned} \frac{\partial rV_0(h|R(h))}{\partial R(h)} &= \frac{-\beta\kappa\lambda s_0(h|R(h)) f_p(h, R(h)) [1 - \Gamma(R(h))]}{r + \delta + \beta\lambda s_0(h, R(h)) [1 - \Gamma(R(h))]} \leq 0 \\ \frac{\partial rV(h, p, p)}{\partial p} &= \frac{(r + \delta) f_p(h, p)}{r + \delta + \beta\lambda s(h, p) [1 - \Gamma(p)]} > 0. \end{aligned}$$

For the case of $\kappa \leq 1$, it trivially follows that the unique reservation productivity level solution is $R(h) = 0$ for any $h \in [0, 1]$. Consider the case where $\kappa > 1$. In this case, inspection of equations (20)

⁵See for example Theorem 2.2 in Boyce and DiPrima (1977)

and (21) reveals that for any h , $V_0(h|b) > V(h, b, b)$ and $V_0(h, 1) < V(h, 1, 1)$. Since $V_0(h|R(h))$ is continuous and everywhere decreasing in $R(h)$ and $V(h, R(h), R(h))$ is continuous and everywhere increasing in $R(h)$ there exists for any $h \in [0, 1]$ a unique $R(h) \in (0, 1)$ that solves equation (19). Hence, it has been shown that there exists a unique steady state equilibrium $s(h, p)$, $s_0(h)$, and $R(h)$. Existence and uniqueness of $w(h, q, p)$ then follows trivially from equation (3) and the asset equations (6) and (8).

The equilibrium condition (16) implies a second kind inhomogeneous Fredholm integral equation in $v(h)$,

$$v(h) - \int_0^1 \psi(h) \frac{\frac{\lambda}{\delta} s_0(h') [1 - \Gamma(R(h'))]}{1 + \frac{\lambda}{\delta} s_0(h) [1 - \Gamma(R(h))]} v(h') dh' = \frac{\psi(h)}{1 + \frac{\lambda}{\delta} s_0(h) [1 - \Gamma(R(h))]}.$$
 (22)

Equation (22) has a degenerate integral kernel which allows the explicit unique solution,

$$\Upsilon(h) = \frac{\int_0^h [1 + \frac{\lambda}{\delta} s_0(h') [1 - \Gamma(R(h'))]]^{-1} d\Psi(h')}{\int_0^1 [1 + \frac{\lambda}{\delta} s_0(h') [1 - \Gamma(R(h'))]]^{-1} d\Psi(h')}.$$
 (23)

Existence of a unique steady state equilibrium solution for $\Upsilon(h)$ then directly follows from the existence and uniqueness of solutions to $s_0(h)$ and $R(h)$.

Existence and uniqueness of the steady state equilibrium unemployment rate u follows directly from equation (13) given existence and uniqueness of $R(h)$, $s_0(h)$, and $\Upsilon(h)$.

Consider pairs of $(h, p) \in [0, 1] \times [0, 1]$ such that $R(h) \geq p$. Then, by definition of the reservation productivity level, $g(h, q, p) = 0$ for any $q \in [0, p]$. Now, consider pairs $(h, p) \in [0, 1] \times [0, 1]$ such that $R(h) < p$. In this case, $g(h, q, p)$ is defined by equation (14). This is a first kind inhomogeneous Volterra integral equation. The proof of existence of a unique solution $g(h, q, p)$ is based on a reformulation of equation (14) into second kind Volterra integral equations. First define $g(h, p) = \int_0^p g(h, q, p) dq$. Equation (14) implies for any $(h, p) \in [0, 1] \times [0, 1]$ such that $R(h) < p$,

$$\int_0^h \int_{R(h')}^p \left[1 + \frac{\lambda}{\delta} [1 - \Gamma(p)] s(h', p') \right] g(h', p') dp' dh' = \frac{\int_0^h s_0(h') [\Gamma(p) - \Gamma(R(h'))] d\Upsilon(h')}{\int_0^1 s_0(h') [1 - \Gamma(R(h'))] d\Upsilon(h')}.$$
 (24)

Differentiation by h and p provides,

$$\begin{aligned} g(h, p) &= \frac{\lambda}{\delta} \int_{R(h)}^p \frac{\gamma(p) s(h, p')}{1 + \frac{\lambda}{\delta} [1 - \Gamma(p)] s(h, p)} g(h, p') dp' \\ &\quad + \frac{\gamma(p) s_0(h) v(h)}{[1 + \frac{\lambda}{\delta} [1 - \Gamma(p)] s(h, p)] \int_0^1 s_0(h') [1 - \Gamma(R(h'))] d\Upsilon(h')}. \end{aligned}$$
 (25)

Equation (25) is a second kind inhomogeneous Volterra integral equation with an everywhere continuous and uniformly bounded integral kernel. Therefore, given existence and uniqueness of $s(h, p)$,

$R(h)$, $s_0(h)$, and $\Upsilon(h)$, there exists a unique solution $g(h, p)$ to equation (25) for $(h, p) \in [0, 1] \times [0, 1]$ such that $R(h) < p$.⁶ For $(h, p) \in [0, 1] \times [0, 1]$ such that $R(h) \geq p$ it trivially obtains that $g(h, p) = 0$. Hence, existence and uniqueness of $g(h, p)$ has been established for all $(h, p) \in [0, 1] \times [0, 1]$. Now, consider $(h, q, p) \in [0, 1]^3$ such that $R(h) < p$ and $R(h) \leq q \leq p$. In this case $g(h, q, p)$ is defined by equation (14). Differentiation of equation (14) with respect to h , q , and p yields,

$$g(h, q, p) - \frac{\lambda}{\delta} \int_{R(h)}^q \frac{\gamma(q)s(h, p)}{1 + \frac{\lambda}{\delta}s(h, p)(1 - \Gamma(q))} g(h, q', p) dq' = \frac{\frac{\lambda}{\delta}s(h, q)\gamma(p)g(h, q)}{1 + \frac{\lambda}{\delta}s(h, p)(1 - \Gamma(q))}. \quad (26)$$

This is a second kind inhomogeneous Volterra integral equation with an everywhere continuous and uniformly bounded integral kernel. Given existence and uniqueness of $s(h, p)$ and $g(h, p)$ a unique solution $g(h, q, p)$ to equation (26) therefore exists for any $(h, q, p) \in [0, 1]^3$ such that $R(h) < p$ and $R(h) \leq q \leq p$. For all other (h, q, p) it follows by definition that $g(h, q, p) = 0$. It has then been shown that a unique steady state equilibrium $g(h, q, p)$ exists. This concludes the proof of existence and uniqueness of steady state equilibrium. ■

3 Properties of steady state equilibrium

In this section I characterize the steady state equilibrium match distribution with a particular focus on sorting. In Shimer and Smith (2000), the analysis of assortative matching is cast in terms of match set characterization. This approach is not informative in my model since agents are willing to match with anybody. Rather, I will present the sorting results in terms of stochastic dominance. Proposition 2 states that assuming $\kappa = 1$ and if the production function is supermodular then the equilibrium firm productivity distribution that a high skill worker is matched with stochastically dominates that of a low skill worker. For the submodular case, the firm productivity distribution of a low skill worker stochastically dominates that of a high skill worker. For the modular case, no sorting results. This is a very strong sorting result. By implication, the worker skill conditional average firm productivity is increasing (decreasing) in skill when the production function is supermodular (submodular).

Define the density of skill h workers that are employed with productivity p firms by,

$$g(h, p) = \int_b^p g(h, q, p) dq. \quad (27)$$

Define the worker skill conditional CDF of firm productivity levels by,

$$\Omega_h(p) = \frac{\int_0^p g(h, p') dp'}{\int_0^1 g(h, p') dp'}. \quad (28)$$

⁶See for example Chapter 2, Theorem 5 in Hochstadt (1973).

With these definitions, Proposition 2 can be stated.

Proposition 2 *For any $h \in [0, 1]$, $\Omega_h(0) = 0$ and $\Omega_h(1) = 1$. Consider any pair $(h_0, h_1) \in [0, 1]^2$ such that $h_0 < h_1$. If $\kappa = 1$ then for all $p \in (0, 1)$,*

- $f_{hp}(h, p) > 0 \forall (h, p) \Rightarrow \Omega_{h_0}(p) > \Omega_{h_1}(p)$ (supermodular).
- $f_{hp}(h, p) < 0 \forall (h, p) \Rightarrow \Omega_{h_0}(p) < \Omega_{h_1}(p)$ (submodular).
- $f_{hp}(h, p) = 0 \forall (h, p) \Rightarrow \Omega_{h_0}(p) = \Omega_{h_1}(p)$ (modular).

The result generalizes to any $\kappa > 0$ as long as $R(h)$ is weakly increasing (decreasing) in h when the production function is supermodular (submodular).

Proof. Define

$$\bar{g}(h, p) = \frac{g(h, p)}{\int_0^1 g(h, p') dp'}.$$

With this, the h conditional firm productivity distribution can be written as,

$$\Omega_h(p) = \int_0^p \bar{g}(h, p') dp', \quad (29)$$

and differentiation of equation (14) with respect to h implies,

$$\int_0^p \left[1 + \frac{\lambda}{\delta} [1 - \Gamma(p)] s(h, p') \right] \bar{g}(h, p') dp' = \frac{\mathbb{I}[R(h) \leq p] [\Gamma(p) - \Gamma(R(h))]}{1 - \Gamma(R(h))}. \quad (30)$$

By continuity of $s(h, p)$ and $\Gamma(p)$ for all $h \in [0, 1]$ and no mass points in $\Gamma(p)$ it follows that for any h , $\bar{g}(h, p)$ is finite and continuous in p . Hence, $\Omega_h(p)$ is continuous in p for any h , and it has no mass points. It immediately follows that for any $h \in [0, 1]$, $\Omega_h(p) = 0$, $\forall p \in [0, R(h)]$. It also follows directly from equation (30) that $\Omega_h(1) = 1$, $\forall h \in [0, 1]$. This establishes Proposition 2 for $p = 1$ and $p \in [0, R(h_0)]$.

To establish the remainder of the proposition, take any pair $(h_0, h_1) \in [0, 1]$ such that $h_0 < h_1$. For the modular case, Lemma 1 states that $s(h_0, p) = s(h_1, p)$. Furthermore by Lemma 2 it follows that $R(h_0) = R(h_1)$. Equation (30) then directly yields $\bar{g}(h_0, p) = \bar{g}(h_1, p)$ for all $p \in [0, 1]$ for the modular case. This establishes Proposition 2 for the modular case.

The remainder of the proof will be shown for the case of supermodularity. The argument can be directly applied to the submodular case with a transformation of the worker skill space such that $\hat{h}(h) = 1 - h$. Thus, assume strict supermodularity of the production function. $\kappa \leq 1$ implies that $R(h) = 0$ for all h . The proof is established for the more general case where $R(h)$ is weakly

increasing in h . Consequently, $R(h_1) \geq R(h_0)$. In contradiction with the claim in Proposition 2, assume for some $p \in (R(h_0), 1)$ that $\Omega_{h_0}(p) \leq \Omega_{h_1}(p)$. By assumption of $p > R(h_0)$, equation (30) implies that $\Omega_{h_0}(p) > 0$. Therefore, it must be that $\Omega_{h_1}(p) \geq \Omega_{h_0}(p) > 0$ and $p > R(h_1)$. Hence, the right hand side of equation (30) must be strictly positive for both h_0 and h_1 .

Differentiation of equation (30) with respect to p yields,

$$\left[1 + \frac{\lambda}{\delta}[1 - \Gamma(p)]s(h, p)\right]\bar{g}(h, p) = \gamma(p) \left[\frac{1}{1 - \Gamma(R(h))} + \frac{\lambda}{\delta} \int_b^p s(h, p')\bar{g}(h, p')dp'\right]. \quad (31)$$

By continuity of $\Omega_h(p)$ in p and by $\Omega_h(0) = 0$ for all h , this implies that there exists some $\hat{p} \in [0, p]$ such that $\Omega_{h_0}(\hat{p}) \leq \Omega_{h_1}(\hat{p})$ and $\bar{g}(h_0, \hat{p}) \leq \bar{g}(h_1, \hat{p})$. To see this, consider first the cases where either $\Omega_{h_0}(p) < \Omega_{h_1}(p)$, or $\Omega_{h_0}(p) = \Omega_{h_1}(p)$ and $g(h_0, p) > g(h_1, p)$. Define the threshold $\underline{p} = \max\{p' \in [0, p] \mid \Omega_{h_0}(p') = \Omega_{h_1}(p')\}$. Existence of \underline{p} is given by $\Omega_h(0) = 0$ for all h . By continuity of $\Omega_h(\cdot)$, it must be that $\Omega_{h_0}(p') \leq \Omega_{h_1}(p')$ for all $p' \in [\underline{p}, p]$. By definition of \underline{p} it must be that, $\int_{\underline{p}}^p \bar{g}(h_0, p')dp' < \int_{\underline{p}}^p \bar{g}(h_1, p')dp'$. Consequently, there must exist some $\hat{p} \in [\underline{p}, p]$ such that $g(h_0, \hat{p}) \leq g(h_1, \hat{p})$. Finally, consider the case where $\Omega_{h_0}(p) = \Omega_{h_1}(p)$ and $g(h_0, p) \leq g(h_1, p)$, then $\hat{p} = p$.

Equation (30) can be re-written as,

$$\Omega(p|h) = 1 - [1 - \Gamma(p)] \left[\frac{1}{1 - \Gamma(R(h))} + \frac{\lambda}{\delta} \int_b^p s(h, p')\bar{g}(h, p')dp'\right]. \quad (32)$$

It then follows that,

$$\begin{aligned} 0 &\geq \Omega(\hat{p}|h_0) - \Omega(\hat{p}|h_1) \\ &= [1 - \Gamma(\hat{p})] \left[\frac{1}{1 - \Gamma(R(h_1))} + \frac{\lambda}{\delta} \int_b^{\hat{p}} s(h_1, p')\bar{g}(h_1, p')dp'\right] - \\ &\quad [1 - \Gamma(\hat{p})] \left[\frac{1}{1 - \Gamma(R(h_0))} + \frac{\lambda}{\delta} \int_b^{\hat{p}} s(h_0, p')\bar{g}(h_0, p')dp'\right], \end{aligned}$$

which implies

$$\frac{1}{1 - \Gamma(R(h_0))} + \frac{\lambda}{\delta} \int_b^{\hat{p}} s(h_0, p')\bar{g}(h_0, p')dp' \geq \frac{1}{1 - \Gamma(R(h_1))} + \frac{\lambda}{\delta} \int_b^{\hat{p}} s(h_1, p')\bar{g}(h_1, p')dp'.$$

By equation (31) and $\bar{g}(h_0, \hat{p}) \leq \bar{g}(h_1, \hat{p})$ one obtains the result,

$$\begin{aligned} \left[1 + \frac{\lambda}{\delta}[1 - \Gamma(\hat{p})]s(h_0, \hat{p})\right]\bar{g}(h_0, \hat{p}) &\geq \left[1 + \frac{\lambda}{\delta}[1 - \Gamma(\hat{p})]s(h_1, \hat{p})\right]\bar{g}(h_1, \hat{p}) \\ &\geq \left[1 + \frac{\lambda}{\delta}[1 - \Gamma(\hat{p})]s(h_1, \hat{p})\right]\bar{g}(h_0, \hat{p}) \\ &\Downarrow \\ s(h_0, \hat{p}) &\geq s(h_1, \hat{p}). \end{aligned}$$

But this contradicts Lemma 1 which dictates that $s(h_0, \hat{p}) < s(h_1, \hat{p})$. Hence, it has been shown that it must be that $\Omega_{h_0}(p) > \Omega_{h_1}(p)$, $\forall p \in (R(h_0), 1)$. ■

The worker skill conditional firm productivity average is given by,

$$E[p|h] = \frac{\int_b^1 p' \hat{g}(h, p') dp'}{\int_b^1 \hat{g}(h, p') dp'} = \int_0^1 [1 - \Omega_h(p')] dp'.$$

Hence, it is a direct implication of Proposition 2 that $E[p|h]$ is strictly increasing in h .

The proof of Proposition 2 highlights that sorting in this model is a result of differential search intensities across worker skill levels. In the supermodular case, positive sorting obtains because more skilled workers search more intensely than less skilled workers for any given point in the firm productivity ladder. As a result, a more skilled worker will end up higher in the ladder in the sense of stochastic dominance. In the submodular case, low skill workers search with greater intensity than high skill workers and the sorting reverses.

The model has an interesting asymmetry in that while for the supermodular case $E[p|h]$ is increasing in h , it is not necessarily true that $E[h|p]$ is everywhere increasing in p . Likewise, the type of stochastic dominance result in Proposition 2 does not necessarily obtain everywhere for the firm productivity conditional worker skill distribution,

$$\Omega_p(h) = \frac{\int_0^h g(h', p) dh'}{\int_0^1 g(h', p) dh'}$$

Given supermodularity, one can show that $\Omega_0(h)$ stochastically dominates $\Upsilon(h)$ and that the overall skill distribution of employed workers dominates $\Omega_0(h)$. However, it is possible that there may exist pairs $p' > p$ where $\Omega_{p'}(h)$ does not stochastically dominate $\Omega_p(h)$.

To understand this result, consider a simple example with two types of workers ($h_0 < h_1$) and three types of firms ($p_0 < p_1 < p_2$). Let the search intensities of the h_0 skill worker be given by, $s(h_0, p_0) = s$ and $s(h_0, p_1) = s(h_0, p_2) = 0$, where $s > 0$. Let the h_1 skill worker's search intensities be given by $s(h_1, p_0) = s(h_1, p_1) = s$ and $s(h_1, p_2) = 0$. Disregard the state of unemployment and assume that the exogenously given job destruction shock moves the worker to the bottom step of the ladder. Hence, this is an example where the high skill worker is searching weakly more intensely at every step of the ladder and strictly more at the middle step. By proposition 2 we know that $E[p|h_0] < E[p|h_1]$ in steady state. But it is straightforward to show that $E[h|p_1] < E[h|p_0] < E[h|p_2]$. Hence, the firm productivity conditional worker skill average is non-monotone in p .

A firm's steady state labor force composition is a result of the skill distribution of the inflow and the skill distribution of the outflow as well as the relative inflow and outflow rates. In the

example above, the skill distribution of the p_1 productivity firms is dominated by that of the p_0 types because the outflow distribution of the p_1 firms dominates that of the p_0 firms and the inflow distribution is the same across the two firm types.

The asymmetry result suggests a potentially important point for empirical studies of sorting: In the study of conditional type distributions, one may obtain qualitatively different results depending on which side of the market one conditions by.

It is worth noting that the sorting results in this paper contrast somewhat with those in Shimer and Smith (2000) where positive assortative matching need not follow from supermodularity, instead requiring the stronger kind of complementarity embodied in log-supermodularity. The difference can be traced to assumptions about search costs and outside options in the two models. In the stopping problem of Shimer and Smith (2000), search costs are related to the passage of time and are consequently worker type dependent as a result of time discounting. Specifically, while the gains to search are greater for more skilled workers given supermodularity of the production function, their search costs are also larger, complicating the final outcome. This complication does not arise in this paper's model since search costs are type independent. In addition, the assumption in Shimer and Smith (2000) of an unmatched income flow that is common to all worker types tends to pull the outcome in the direction of negative sorting. For example, for the modular case, the gains of matching a high type worker exceed that of a low type. Consequently, a social planner would attempt to ensure that high type agents are quickly matched by maintaining a pool of unmatched low type agents which stay unmatched because they are refusing to match with each other, instead waiting for a high type to come along. This will generate negative sorting even though the production function is modular. In this paper's model, this effect would be reflected in a productivity threshold $R(h)$ that is decreasing in h . If instead, as assumed in this paper, unmatched agents receive income equal to what they receive if matched with the lowest type, the pull toward negative assortative matching disappears.

4 Two simple examples

The following presents results for two simple cases that differ only in the specification of the production function. The first example assumes a supermodular production function $f(h, p) = (h+1)(p+1)$. The second example assumes log-supermodularity with a particular parameterization that highlights the asymmetry discussed above between the Ω and Ω_p distributions. In this case the production function is specified as $f(h, p) = 5 \exp(\ln(h+1) \ln(p+1))$. As in Christensen, Lentz,

Mortensen, Neumann, and Werwatz (2005), the search cost function is specified by,

$$c(s) = c_0 \frac{s^{1+\frac{1}{\gamma}}}{1+\frac{1}{\gamma}}. \quad (33)$$

By equation (10), one then obtains a simple equation for $\bar{s}(h, p) = \lambda\beta s(h, p)$,

$$\bar{s}(h, p) = \alpha \left(\int_p^1 \frac{f'_p(h, p') (1 - \Gamma(p')) dp'}{r + \delta + \bar{s}(h, p') (1 - \Gamma(p'))} \right)^\gamma, \quad \forall p \geq b, \quad (34)$$

where $\alpha = (\beta\lambda)^{1+\gamma}/c_0^\gamma$. Furthermore, assume $\kappa = 1$ and make the following specifications,

$$\lambda = 0.25$$

$$\beta = 0.50$$

$$\gamma = 5.00$$

$$c_0 = 0.10$$

$$r = 0.05$$

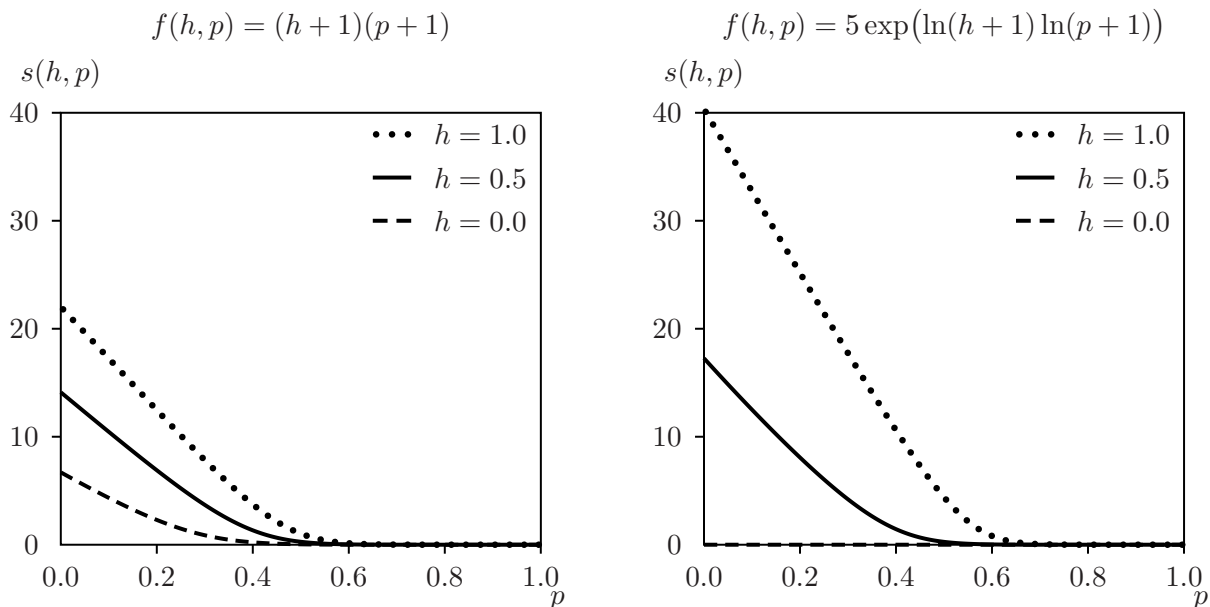
$$\delta = 0.25$$

Assume that both $\Gamma(\cdot)$ and $\Upsilon(\cdot)$ are uniform on support $[0, 1]$. Both $\Gamma(\cdot)$ and $\Upsilon(\cdot)$ are of course endogenous to the steady state equilibrium, but it is straightforward to map back to $\Psi(\cdot)$ and $\Phi(\cdot)$.

Figure 1 shows the search intensity solutions for both cases. It is seen that the search intensities are monotonically decreasing in p and increasing in h , confirming Lemma 1. The somewhat stronger complementarities in production in the log-supermodular case will tend to produce a greater spread in search intensities across worker types.

Figure 2 shows the conditional type distributions and averages for the case of $f(h, p) = (h + 1)(p + 1)$. The left column shows the worker skill conditional firm productivity distributions and averages. It is seen that the stochastic dominance result in Proposition 2 is confirmed. Consequently, the average firm type $E[p|h]$ is monotonically increasing in h . The right column presents the firm type conditional worker distributions and averages. Here, it is seen that the worker skill distribution of higher type firms does not everywhere stochastically dominate that of less productive firms. Specifically, $\Omega_{0,0}(h)$ is not stochastically dominated by $\Omega_{0,3}(h)$. While not easily seen in the figure, $\Omega_{0,3}(h) < \Omega_{0,0}(h)$ for sufficiently small h . As h increases $\Omega_{0,3}(h)$ becomes greater than $\Omega_{0,0}(h)$. Thus, for the lower range of firm productivity levels, as firm productivity increases, the mass of low skill workers does in fact decrease in firm productivity, but so does the mass of high

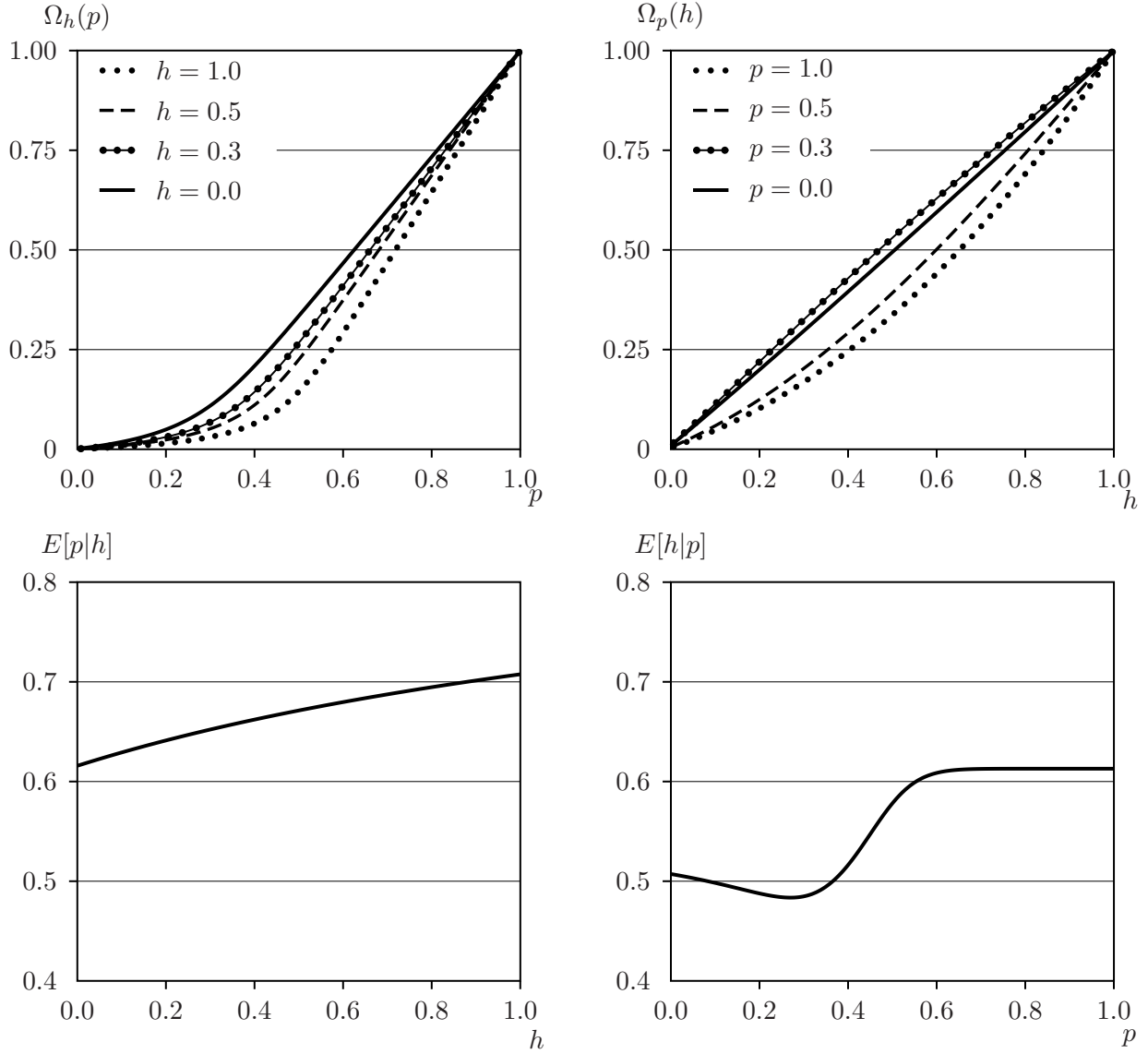
Figure 1: Search intensity



skilled workers with the result that the average skill level is decreasing in firm productivity over this particular range. Eventually, the positive assortative matching does push through to yield that for sufficiently high firm productivity levels, the worker skill distribution stochastically dominates that of the low productivity firms. In particular, the worker skill distribution of the highest productivity firm stochastically dominates that of all other firm productivity levels. Hence, the average worker type conditional on firm type $E[h|p]$ is non-monotonic in p , first declining and then increasing to reach a maximum at the highest firm productivity level. This property will be repeated more clearly in Figure 3.

Turning to the log-supermodular case in Figure 3, it is seen that the properties of the example in Figure 2 are simply amplified in face of the stronger production function complementarities.

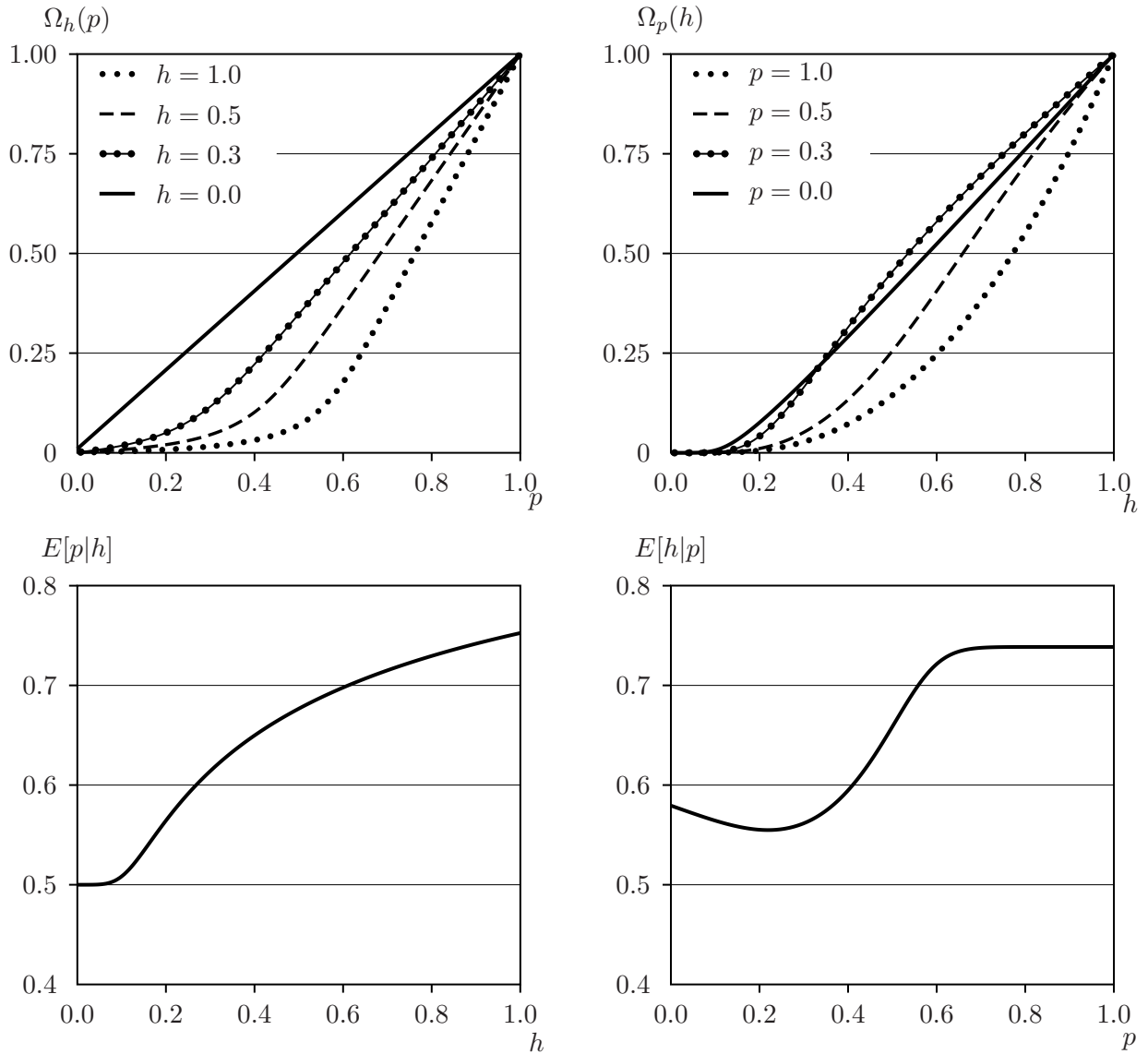
Figure 2: Conditional type distributions and averages for $f(h, p) = (h + 1)(p + 1)$.



5 Conclusion

This paper presents an equilibrium model where workers choose search intensity on and off the job to generate outside employment opportunities. Wages are determined endogenously in equilibrium through bargaining where outside job offers can impact the threat points of the bargaining game. It is shown that given equal search efficiency on and off the job, supermodularity (submodularity) of the production function implies that the conditional firm productivity distribution is stochastically increasing (decreasing) in worker skill. By implication the average firm productivity is increasing

Figure 3: Conditional type distributions and averages for $f(h, p) = 5 \exp(\ln(h + 1) \ln(p + 1))$.



(decreasing) in worker skill given supermodularity (submodularity).

Interestingly the model contains an important asymmetry in that the reverse conditioning is not necessarily characterized by a similar stochastic dominance result. Specifically, it need not be that the skill distribution of workers matched with a more productive firm stochastically dominates that of a less productive firm. This suggests a potentially important complication in the empirical measurement of sorting in the labor market.

The sufficient conditions for sorting in this model differ somewhat from that in Shimer and

Smith (2000) where for example supermodularity is not sufficient to generate positive assortative matching.

A Detailed derivations

Consider an employed worker of type h who is employed with a type p firm at employment contract (w, s) . Denote by $q = q(h, w, p)$, the threshold type such that a meeting of an outside firm with type less than q has no impact on the worker's wage. Furthermore, adopt the short hand $V(h, q, p)$ as the value of employment to a type h worker who is employed with a type p firm subject to an employment contract set through bargaining where the worker had the threat point to accept outside employment with a type q firm. The value function, $\tilde{V}(h, p, w, s)$, for the employed worker is,

$$\begin{aligned}
(r + \delta)\tilde{V}(h, p, w, s) &= w - c(s) + \lambda s \int_p^1 [V(h, p, p') - V(h, q, p)] d\Gamma(p') + \\
&\quad \lambda s \int_q^p [V(h, p', p) - V(h, q, p)] d\Gamma(p') + \delta V_0(h) \\
&= w - c(s) + \delta V_0(h) - \lambda s(1 - \Gamma(q))V(h, q, p) + \\
&\quad \lambda s \int_p^1 [\beta V(h, p', p') + (1 - \beta)V(h, p, p)] d\Gamma(p') + \\
&\quad \left. \lambda s \int_q^p [\beta V(h, p, p) + (1 - \beta)V(h, p', p')] d\Gamma(p') \right\} \\
&= w - c(s) + \delta V_0(h) - \lambda s(1 - \Gamma(q))V(h, q, p) + \\
&\quad \lambda s(1 - \beta)(1 - \Gamma(p))V(h, p, p) + \lambda s\beta(1 - \Gamma(p))V(h, p, p) + \\
&\quad \lambda s\beta \int_p^1 (1 - \Gamma(p'))V_p(h, p', p') dp' + \\
&\quad \lambda s\beta(\Gamma(p) - \Gamma(q))V(h, p, p) - \lambda s(1 - \beta)(1 - \Gamma(p))V(h, p, p) + \\
&\quad \left. \lambda s(1 - \beta)(1 - \Gamma(q))V(h, q, q) + \lambda s(1 - \beta) \int_q^p (1 - \Gamma(p'))V_p(h, p', p') dp' \right\} \\
&= w - c(s) + \delta V_0(h) - \lambda s(1 - \Gamma(q))V(h, q, p) + \\
&\quad \lambda s\beta(1 - \Gamma(q))V(h, p, p) + \lambda s(1 - \beta)(1 - \Gamma(q))V(h, q, q) + \\
&\quad \lambda s\beta \int_p^1 (1 - \Gamma(p'))V_p(h, p', p') dp' + \lambda s(1 - \beta) \int_q^p (1 - \Gamma(p'))V_p(h, p', p') dp' \Big\} \\
&= w - c(s) + \delta V_0(h) + \lambda s\beta \int_p^1 \frac{f_p(h, p')(1 - \Gamma(p')) dp'}{r + \delta + \beta \lambda s(h, p')(1 - \Gamma(p'))} + \\
&\quad \left. \lambda s(1 - \beta) \int_q^p \frac{f_p(h, p')(1 - \Gamma(p')) dp'}{r + \delta + \beta \lambda s(h, p')(1 - \Gamma(p'))} \right\},
\end{aligned}$$

where the last equality follows from applying the envelope theorem to the equation,

$$(r + \delta)V(h, p, p) = \max_{s \geq 0} \left\{ f(h, p) - c(s) + \delta V_0(h) + \lambda s\beta \int_p^1 [V(h, x, x) - V(h, p, p)] d\Gamma(x) \right\}.$$

This yields,

$$\begin{aligned}
(r + \delta)V_p(h, p, p) &= f'_p(h, p) - \lambda s(h, p)\beta(1 - \Gamma(p))V_p(h, p, p) \\
&\Downarrow \\
V_p(h, p, p) &= \frac{f_p(h, p)}{r + \delta + \beta\lambda s(h, p)(1 - \Gamma(p))}.
\end{aligned}$$

B Employment contract bargaining

At the beginning of an employment relationship, the firm and the worker bargain over a constant wage and worker's search intensity that will remain in effect until the relationship terminates or both parties consent to renegotiation. The bargaining game is an application of the alternating offers game of Rubinstein (1982) and most resembles the exogenous break down version as presented in Binmore, Rubinstein, and Wolinsky (1986). The following two subsections present the subgame perfect equilibrium for the case of an unemployed worker, and an employed worker who is renegotiating subsequent to an outside offer, respectively. The arguments are closely related to the bargaining games described in Cahuc, Postel-Vinay, and Robin (2006), although the bargaining is simplified to take place in artificial time with zero disagreement values and elimination of the possibility of meeting another employer during bargaining.

The outcomes of the alternating offers games are identical to that of axiomatic Nash bargaining where the threat point of the firm is always zero for the firm, and the worker's threat point is either unemployment or full surplus extraction from the least productive of the two firms competing over the worker.⁷ Specifically, the bargaining outcome of an unemployed worker maximizes the Nash product,

$$\{w_0(h, p), s(h, p)\} = \arg \max_{w, s} (\tilde{V}(h, p, w, s) - V_0(h))^\beta \tilde{J}(h, w, p, s)^{(1-\beta)}, \quad (35)$$

which yields the worker valuation,

$$V(h, R(h), p) = \beta V(h, p, p) + (1 - \beta)V_0(h). \quad (36)$$

The inclusion of the reservation productivity argument implicitly states that the worker will only accept to bargain with employer types greater than $R(h)$.

The outcome of a worker bargaining with two employer types, q and p such that $p > q$ is that the worker will negotiate an employment contract with the type p firm with a threat point of full

⁷Dey and Flinn (2005) assume this axiomatic Nash bargaining protocol.

surplus extraction and efficient search intensity with the lower type firm, $V(h, q, q)$. Hence, the employment contract that results from this bargaining setting is,

$$\{w(h, q, p), s(h, p)\} = \arg \max_{w, s} (\tilde{V}(h, p, w, s) - V(h, q, q))^\beta \tilde{J}(h, w, p, s)^{(1-\beta)}. \quad (37)$$

The bargaining outcome is,

$$V(h, q, p) = \beta V(h, p, p) + (1 - \beta)V(h, q, q). \quad (38)$$

In both cases (35) and (37), the agreed upon search intensity $s(h, p)$ is the one that maximizes total match surplus, $V(h, p, p)$. This is the jointly efficient search intensity level and does not depend on the specific surplus split dictated by bargaining power and threat points.

B.1 Unemployed worker

Consider an alternating offers game where the worker makes an offer (w_e, s_e) to the firm. If the firm accepts, employment starts and the worker receives payoff $\tilde{V}(h, p, w_e, s_e)$ and the firm receives $\tilde{J}(h, p, w_e, s_e) = \tilde{V}(h, p, f(h, p), s_e) - \tilde{V}(h, p, w_e, s_e)$. If the firm rejects the offer, the bargaining breaks down with exogenous probability Δ . If so, the firm receives a zero payoff and the worker goes back to unemployment and receives $V_0(h)$. If bargaining does not break down, the bargaining moves to the next round where the firm makes an offer (w_f, s_f) with probability $1 - \beta$ and the worker gets to make the offer (w_e, s_e) with probability β . If the firm makes the offer and the worker accepts, the worker receives $\tilde{V}(h, p, w_f, s_f)$ and the firm receives $\tilde{J}(h, p, w_f, s_f) = \tilde{V}(h, p, f(h, p), s_f) - \tilde{V}(h, p, w_f, s_f)$. If the worker rejects, the game moves on to the next round if no break down occurs. And again, the worker will make the offer with probability β and the firm with probability $1 - \beta$. The game continues like this ad infinitum or until agreement is reached. Disagreement payoffs are zero and the discount rate between rounds is zero.

Both the worker and the firm will offer the same search intensity, $s_e = s_f = s(h, p)$, where $s(h, p) = \arg \max_s \tilde{V}(h, p, f(h, p), s)$. Furthermore, consider the strategies where the worker accepts any offer (w, s) such that $\tilde{V}(h, p, w, s) \geq \tilde{V}(h, p, w_f, s(h, p))$ and rejects any offer such that $\tilde{V}(h, p, w, s) < \tilde{V}(h, p, w_f, s(h, p))$. Similarly, the firm accepts any offer (w, s) such that $\tilde{J}(h, p, w, s) \geq \tilde{J}(h, p, w_e, s(h, p))$ and rejects any offer such that $\tilde{J}(h, p, w, s) < \tilde{J}(h, p, w_e, s(h, p))$.

By definition the firm's payoff satisfies $\tilde{J}(h, p, w, s) = \tilde{V}(h, p, f(h, p), s) - \tilde{V}(h, p, w, s)$. Hence, a firm accepts any offer such that

$$\tilde{V}(h, p, w, s) \leq \tilde{V}(h, p, w_e, s(h, p)) - \tilde{V}(h, p, f(h, p), s(h, p)) + \tilde{V}(h, p, f(h, p), s). \quad (39)$$

It is seen that the right hand side of the firm acceptance condition (39) is maximized for $s = s(h, p)$ and does not depend on w . Hence, any worker deviation $s'_e \neq s_e = s(h, p)$ that will be accepted by the firm must result in a worker payoff $\tilde{V}(h, p, w, s'_e) < \tilde{V}(h, p, w_e, s(h, p))$, for any w , which is not profitable.

A similar argument can be made that the firm will not want to deviate from $s_f = s(h, p)$. The worker will accept any offer such that,

$$\tilde{J}(h, p, w, s) \leq \tilde{V}(h, p, f(h, p), s) - \tilde{V}(h, p, w_f, s(h, p)). \quad (40)$$

It is seen that the right hand side of the worker acceptance decision (40) is maximized for $s = s(h, p)$ and that it does not depend on w . Hence, any firm deviation $s'_f \neq s_f = s(h, p)$ that will be accepted by the worker must result in a firm payoff $\tilde{J}(h, p, w, s'_f) < \tilde{J}(h, p, w_f, s_f)$, for any w , which is not profitable.

It also follows directly from the above acceptance arguments that any strategy that prescribes $s_e \neq s(h, p)$ or $s_f \neq s(h, p)$ cannot be an equilibrium because a deviation to $s(h, p)$ will be profitable.

Now consider potential deviations in the wage. The worker's payoff $\tilde{V}(h, p, w, s_e)$ is monotonically increasing in w . It follows directly from (39) that any worker wage offer deviation w'_e that will be accepted by the firm is such that $w'_e \leq w_e$. This is not profitable. Any other deviation will not be accepted by the firm and is therefore also not profitable. A similar argument applies to possible firm wage offer deviations.

Subgame perfection of the acceptance strategies requires that the worker is indifferent between accepting the firm's offer (w_f, s_f) and rejecting it. A similar indifference applies on the firm side. This disciplines the acceptance levels by,

$$\hat{V}(w_f) = (1 - \Delta)[\beta\hat{V}(w_e) + (1 - \beta)\hat{V}(w_f)] + \Delta V_0(h) \quad (41)$$

$$\hat{J}(w_e) = (1 - \Delta)[\beta\hat{J}(w_e) + (1 - \beta)\hat{J}(w_f)] \quad (42)$$

where $\hat{V}(w) = \tilde{V}(h, p, w, s(h, p))$ and $\hat{J}(w) = \tilde{J}(h, p, w, s(h, p))$. Equations (41) and (42) can be rewritten as,

$$\beta[\hat{V}(w_f) - \hat{V}(w_e)] = \Delta[V_0(h) - \beta\hat{V}(w_e) - (1 - \beta)\hat{V}(w_f)] \quad (43)$$

$$(1 - \beta)[\hat{J}(w_f) - \hat{J}(w_e)] = \Delta[\beta\hat{J}(w_e) + (1 - \beta)\hat{J}(w_f)]. \quad (44)$$

Taking the limit as $\Delta \rightarrow 0$, equations (41) and (42) imply that $w_f \rightarrow w_e$. Denote the common

limit by w . Hence,

$$\begin{aligned}\frac{\partial \hat{V}(w)}{\partial w} &= \lim_{\Delta \rightarrow 0} \frac{\hat{V}(w_f) - \hat{V}(w_e)}{w_f - w_e} \\ \frac{\partial \hat{J}(w)}{\partial w} &= \lim_{\Delta \rightarrow 0} \frac{\hat{J}(w_f) - \hat{J}(w_e)}{w_f - w_e}.\end{aligned}$$

Since changes in w only affect the match surplus split, it follows that $\partial \hat{V}(w)/\partial w = -\partial \hat{J}(w)/\partial w$. Hence, taking the limit $\Delta \rightarrow 0$ in equations (43) and (44) yields,

$$\begin{aligned}-\frac{\beta}{1-\beta} &= \frac{V_0(h) - \beta \hat{V}(w) - (1-\beta)\hat{V}(w)}{\beta \hat{J}(w) + (1-\beta)\hat{J}(w)} \\ &\Downarrow \\ \hat{V}(w) &= \beta \hat{V}(f(h, p)) + (1-\beta)V_0(h).\end{aligned}\tag{45}$$

Hence, as the break down probability goes to zero, the outcome of the alternating offers game limits to the outcome of the axiomatic Nash bargaining outcome in equation (36).

B.2 Employed worker

Cahuc, Postel-Vinay, and Robin (2006) provide a strategic bargaining foundation for the axiomatic Nash bargaining outcome in equation (38). The outcome is a subgame perfect equilibrium in a game based on firms submitting bids for the worker subject to a worker's option to use the bids as threat points in a subsequent strategic bargaining game identical to the one described in the previous section. In the game between two employers of types q and p , respectively, where $q \leq p$, the higher type firm wins by submitting a contract bid $(w, s(h, p))$ as stated in equation (38).

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