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PRODUCTIVITY GROWTH AND WORKER REALLOCATION*

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Productivity dispersion across firms is large and persistent, and worker reallocation among firms is an important source of productivity growth. An equilibrium model of growth and firm evolution designed to clarify the role of worker reallocation in the growth process is studied. We show that it explains the correlations between size measures and labor productivity found in Danish firm data. Conditions under which the reallocation of workers from less to more productive firms contributes to aggregate productivity growth in the economy modeled are derived. Finally, a proof of existence of an equilibrium solution to the model is also provided.

1. INTRODUCTION

In their review article on firm productivity, Bartelsman and Doms (2000) draw three lessons from the empirical studies based on longitudinal plant and firm data. First, the extent of dispersion in productivity across production units, firms, or establishments is large. Second, the productivity rank of any unit in the distribution is highly persistent. Third, a large fraction of aggregate productivity growth is the consequence of worker reallocation across firms.

Although the explanations of productive heterogeneity across firms are not fully understood, economic principles suggest that its presence should induce worker reallocation from less to more productive firms as well as from exiting to entering firms. Indeed, more productive employers should have a profit incentive to expand production. There is ample evidence that workers do flow from one firm to another frequently. As Davis et al. (1996) and others document, job and worker flows are large, persistent, and essentially idiosyncratic in the United

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States. Recently, Fallick and Fleischman (2001) and Stewart (2002) have found that job-to-job flows without a spell of unemployment in the United States represent at least half of the separations and is growing. In their analysis of Danish matched employer–employee data, Frederiksen and Westergaard-Nielsen (2002) report that the average annual establishment separation rate over the 1980–1995 period was 26%. About two-thirds of the outflow represents the movement of workers from one firm to another.

This article starts with the observation that there is no correlation between labor force size and labor productivity in Danish firm-level data but a strong positive association between value added and labor productivity. These relationships are inconsistent with a standard competitive model with total factor productivity (TFP) differences across firms. In this article, we show that an extension of the alternative model developed by Klette and Kortum (2004) can explain the facts. Their model was designed to be consistent with stylized facts about product innovation and its relationship to the dynamics of firm size evolution. Because it is based on the endogenous growth model of Grossman and Helpman (1991), our amended version also offers possible insight into the role of worker reallocation as a source of equilibrium productivity growth.

In the model, firms are monopoly suppliers of differentiated intermediate products that serve as inputs in the production of a final consumption good. Cheaper and better-quality products are introduced from time to time as the outcome of R&D investment by both existing firms and new entrants. As new products and services displace old ones, a process of creative destruction induces the need to reallocate workers across productive activities engaged in by firms. In the version of the model studied here, better products and services require less labor input, and a firm's current productivity can depend on the number and quality of its past product innovations in general.

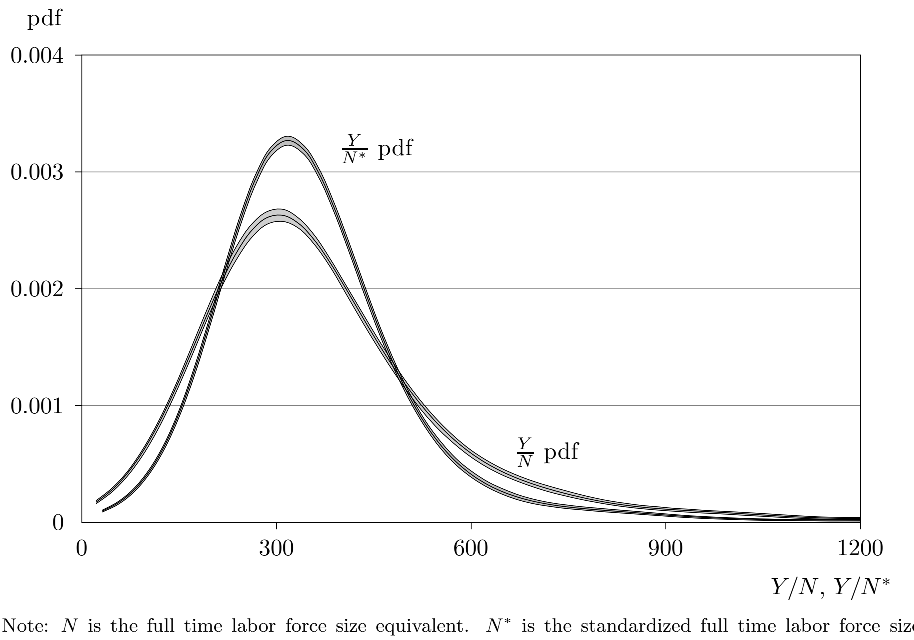
As a theoretical result, we show that more productive firms, those that have developed higher-quality products in the past, tend to grow larger by developing more product lines in the future only if a firm's future product quality is positively correlated with those developed in the past. Conversely, if the expected quality of any future product line is identical across firms, then investment in R&D is independent of a firm's current productivity. Interestingly, the qualitative relationship between employment size and labor productivity is ambiguous in the first case and is negative in the second because innovations are labor saving in the model. However, more productive firms can be expected to develop more product lines and enjoy larger sales volume if persistent differences in the quality of intermediate products and services supplied exist. If more productive firms do grow faster in this sense, then aggregate productivity growth reflects the fact that workers flow from less to more productive employers as well as from exiting to entering firms. Finally, we prove that an equilibrium solution exists to the model when there are persistent differences in firm productivity.

The remainder of the article is organized as follows: In Section 2, we take a brief look at the correlations between value added and employment found in Danish firm panel data. The third section describes the model. The concept of an equilibrium solution to the model is defined and a proof of existence is provided in the fourth section. The implications of the model for aggregate labor reallocation are reviewed in Section 5. The article concludes with a brief statement of the results found in the article and discussion of future work.

2. DANISH FIRM DATA

Danish firm data provide information on productivity dispersion and the relationships among productivity, employment, and value added. The available data set is an annual panel of privately owned firms for the years 1992–1997 drawn from the Danish Business Statistics Register. The sample of approximately 6700 firms is restricted to those with 20 or more employees. The variables observed in each year include value added (Y), the total wage bill (W), and full-time equivalent employment (N). In this article we use these relationships to motivate the theoretical model studied. Both Y and W are measured in 1000 Danish Kroner and N is simply a body count.

Nonparametric estimates of the distributions of two alternative measures of a firm's labor productivity are illustrated in Figure 1. In each case, the middle line represents point estimates and the upper and lower lines denote the boundaries of a 90% confidence interval. The first measure is value added per worker (Y/N) whereas the second is valued added per unit of quality-adjusted employment



Note: N is the full time labor force size equivalent. N^* is the standardized full time labor force size equivalent. The shaded areas represent 90% confidence bounds.

FIGURE 1

VALUE ADDED PER WORKER PDF. PRIVATELY OWNED DANISH FIRMS, 1992

(Y/N^*). The first measure misrepresents cross-firm productivity differences to the extent that labor quality differs across firms. However, if more productive workers are compensated with higher pay as would be true in a competitive labor market, one can use a wage-weighted index of employment to correct for this source of cross-firm differences in productive efficiency. Formally, the constructed quality-adjusted employment of firm j is defined as $N_j^* = W_j/w$ where $w = \sum_j W_j/(\sum_j N_j)$ is the average wage paid per worker in the market. Although correcting for wage differences across firms in this manner does reduce the spread and skew of the implied productivity distribution somewhat, both distributions have high variance and skew and are essentially of the same shape.

Both distributions are consistent with those found in other data sets (see Bartelsman and Doms, 2000). For example, the distribution is skewed to the right and is very dispersed. In the case of the adjusted measure of productivity, the 5th percentile is roughly half the mode whereas the 95th percentile is approximately twice as large as the mode. The range between the two represents a fourfold difference in value added per worker across firms.

There are many potential explanations for the cross-firm productivity differentials. A comparison of the two density estimates in Figure 1 suggests that differences in the quality of labor inputs do not seem to be essential. For example, TFP differences across firms can be expected as a consequence of slow diffusion of new techniques. If technical improvements were factor neutral and output markets were perfectly competitive, then one would expect that more productive firms would acquire more labor and more capital. The implied consequence would seem to be a positive relationship between labor force size and labor productivity. Interestingly, there is no correlation between the two in Danish data.

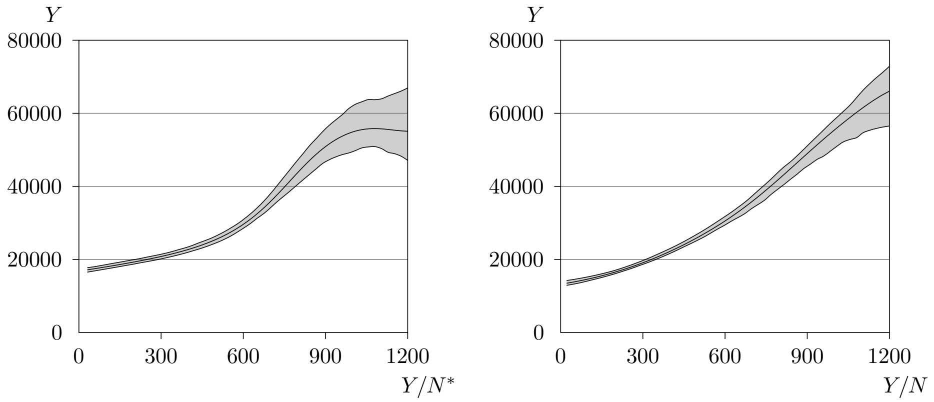
The correlations between the two measures of labor productivity with the two employment measures and sales as reflected in value added are reported in Table 1. As documented in the table, the correlation between labor force size and productivity using either the raw employment measure or the adjusted one is zero. However, note the strong positive association between value added and both measures of labor productivity. Figure 2 illustrates nonparametric regressions of value added on the two productivity measures. The top and bottom curves in the figures represent the boundaries of a 90% confidence interval for the relationship. The positive relationships illustrated in the figures are highly significant.

The theory developed in this article is motivated by these observations. Specifically, it is a theory that postulates labor-saving technical progress of a specific form. Hence, the apparent fact that more productive firms produce more with roughly the same labor input per unit is consistent with the model.

TABLE 1

PRODUCTIVITY–SIZE CORRELATIONS

	Employment (N)	Adjusted Employment (N^*)	Value Added (Y)
Y/N	0.0331	0.1397	0.3944
Y/N^*	0.0114	−0.0076	0.2618



Note: The shaded areas represent 90% bootstrap confidence intervals.

FIGURE 2

VALUE ADDED PER WORKER REGRESSED AGAINST VALUE ADDED, 1992

3. A MODEL OF GROWTH THROUGH CREATIVE DESTRUCTION

As is well known, firms come in an amazing range of shapes and sizes. This fact cannot be ignored in any analysis of the relationship between firm size and productivity. Furthermore, an adequate theory must account for entry, exit, and firm evolution in order to explain the size distributions observed. Klette and Kortum (2004) construct a stochastic model of firm product innovation and growth that is consistent with stylized facts regarding the firm size evolution and distribution. The model also has the property that technical progress is labor saving. For these reasons, we pursue their approach in this article.

Although Klette and Kortum allow for productive heterogeneity across firms, firm productivity and growth are unrelated in their specification because costs and benefits of growth are both proportional to firm productivity. In our version of the model, this outcome is a special case of a more general formulation in which productivity is a stochastic characteristic of new products. Allowing for a positive relationship between firm growth rates and firm productivity is necessary for consistency with the relationships found in the Danish firm data. Finally, the model is one of dynamic general equilibrium with important implications about the role of reallocation as a source of aggregate productivity growth.

3.1. *Preferences and Technology.* Intertemporal utility of the representative household at time t is given by

$$U_t = \int_t^{\infty} \ln C_s e^{-\rho(s-t)}\, ds \tag{1}$$

where $\ln C_t$ denotes the instantaneous utility of the single consumption good at date t and ρ represents the pure rate of time discount. Each household is free to borrow or lend at interest rate r_t. Nominal household expenditure at date t is $E_t = P_t C_t$. Optimal consumption expenditure must solve the differential equation $\dot{E}/E = r_t - \rho$. Following Grossman and Helpman (1991), we choose the numeraire so that $E_t = 1$ for all t without loss of generality, which in turn implies $r_t = \rho$ for all t. Note that this choice of the numeraire also implies that the price of the consumption good, P_t, falls over time at a rate equal to the rate of growth in consumption.

The quantity of the consumption good produced is determined by the quantity and productivity of the economy's intermediate inputs. Specifically, there is a unit continuum of inputs and consumption is determined by the production function

$$\ln C_t = \int_0^1 \ln(A_t(j)x_t(j))\,dj = \ln A_t + \int_0^1 \ln x_t(j)\,dj \tag{2}$$

where $x_t(j)$ is the quantity of input $j \in [0, 1]$ at time t, $A_t(j)$ is the productivity of input j at time t, and

$$\ln A_t \equiv \int_0^1 \ln A_t(j)\,dj$$

represents the log of aggregate productivity. The level of productivity of each input is determined by the aggregate number of technical improvements made in the past. Specifically,

$$A_t(j) = \Pi_{i=1}^{J_t(j)} q_i(j)$$

where $J_t(j)$ is the number of innovations made in input j up to date t and $q_i(j) > 1$ denotes the relative improvement (step size) in productivity attributable to the ith innovation. Innovations arrive at rate δ, which is endogenous and the same for all intermediate products.

The model is constructed so that a steady-state growth path exists with the following properties: Consumption output grows at a constant rate whereas the quantities of intermediate products and the endogenous innovation frequency are stationary and identical across all intermediate goods. As a consequence of the law of large numbers, the assumption that the number of innovations to date is Poisson with arrival frequency δ for all intermediate goods and the assumption that the relative productivity improvement of an innovation is independent of good type and innovation number imply

$$\begin{aligned} \ln C_t &= \ln A_t + \int_0^1 \ln x(j)\,dj = \int_0^1 \sum_{i=1}^{J_t(j)} \ln q_i(j)\,dj + \int_0^1 \ln x(j)\,dj \\ &= \delta t\, Eq + \int_0^1 \ln x(j)\,dj, \quad \text{where} \quad Eq = \int_0^1 \ln q_i(j)\,dj \quad \text{for all } i \end{aligned} \tag{3}$$

which represents the expected relative improvement in the productivity of intermediate goods. In other words, the rate of growth in consumption and productivity is the product of the creative–destruction rate and the expected log of the productivity improvement step size induced by the typical new innovation.

3.2. *Firm Competition.* Each individual firm is the monopoly supplier of the products it created in the past that have survived to the present. The price charged for each is limited by the ability of suppliers of previous versions to provide a substitute. In the Nash–Bertrand equilibrium, any innovator takes over the market for its good type by setting a limit price just below that at which consumers are indifferent between the higher-quality product supplied by the innovator and an alternative supplied by the last provider. The price charged is the product of the relative quality and the previous producer's marginal cost of production.

Labor is the only factor in the production of intermediate inputs. Labor productivity is the same across all inputs and is set equal to unity. Hence, the limit price $p = qw$ is the product of the wage cost, w, which represents the marginal cost of production of the incumbent supplier, and the relative improvement in the quality of the innovators product, q. Note that there are two equivalent interpretations of the model. Either the innovator's version of the intermediate good is q times more productive in the production of the consumption good as just suggested or the innovator can provide the same good that is much cheaper than the incumbent.[2]

As total expenditure is normalized at unity and there is a unit measure of product types, it follows that total revenue per product type is also unity, i.e., $px = 1$. Hence, product output and employment are both equal to

$$x = \frac{1}{p} = \frac{1}{wq} \tag{4}$$

and the gross profit associated with supplying the good is

$$1 > \pi = px - wx = 1 - \frac{1}{q} > 0 \tag{5}$$

The labor-saving nature of improvements in intermediate input quality is implicit in the fact that labor demand is decreasing in q.

Following Klette and Kortum, the discrete number of products supplied by a firm, an integer denoted as k, evolves over time as a birth–death process reflecting product creation and destruction. In their interpretation, k reflects the number of the firm's past successes in the innovation process as well as its current size. New products and services are generated by R&D investment. The firm's R&D investment flow generates new product arrivals at frequency γk. The total R&D

[2] Given the symmetry of demands for the different good types and the assumption that future quality improvements are independent of the type of good, one can drop the good subscript without confusion. Given stationarity of quantities along the equilibrium growth path, the time subscript can be dropped as well.

investment cost is $wc(\gamma)k$ where $c(\gamma)k$ represents the labor input required in the R&D process. The function $c(\gamma)$ is assumed to be strictly increasing and convex. According to the authors, the implied assumption that the total cost of R&D investment is linearly homogenous in the new product arrival rate and the number of existing product, "captures the idea that a firm's knowledge capital facilitates innovation."

It should be kept in mind that the number of "products" provided can be broadly interpreted. Of course, it includes the case of, say, an auto firm that has grown by developing a series of different model lines. In this case, a more productive firm, say Toyota, succeeds by developing higher-quality cars and trucks. But, it also includes a Walmart. In this type of case, the firm produces retail services more cheaply and grows by replacing less efficient retailers in many local markets.

The market for any current product supplied by any firm is destroyed by the creation of a new version by some other firm, which occurs at rate δ. Below we refer to γ as the firm's creation rate and to δ as the common destruction rate faced by all firms. As product gross profit and product quality are one to one, the profits earned on each product reflect a firm's current labor productivity. The firm chooses the creation rate γ to maximize the expected present value of its future net profit flow conditional on information that is relevant for predicting the quality of future innovations.

3.3. *The Value of a Firm.* Let the parameter vector θ summarize past realizations of π. We assume that this indicator is a sufficient statistic for the distribution of the next innovation's profit. For example, the product quality sequence might be a first-order Markov process, in which case θ is the profit on the last product innovation. Alternatively, we might think of the problem as one in which a firm's product profitability is initially unknown but can be learned over time by observing the past realization. In the normal–normal case, the sufficient statistic is a pair that includes both the current estimate of the mean and its precision. In general, θ will be updated in response to the realized profitability of any new product.

Let $\Pi^k = (\pi_1, \pi_2, \dots, \pi_k)$ denote the firm's vector of profits for the products currently supplied, let $\Pi^{k+1} = (\Pi^k, \pi')$ represent the profits of the $k+1$ products where $\pi_{k+1} = \pi'$, and let $\Pi^k_{\langle i\rangle}$ denote Π^k excluding element $i \in \{1, \dots, k\}$. In terms of this notation, the current value of the firm is a function of its state characterized by Π^k and θ. It solves the Bellman equation

(6)

$$rV_k(\Pi^k, \theta) = \max_{\gamma \geq 0} \Bigg\{ \sum_{i=1}^{k} \pi_i - wc(\gamma)k + \gamma k \{E[V_{k+1}((\Pi^k, \pi'), \theta') \mid \theta] - V_k(\Pi^k, \theta)\} + \delta k \Bigg[\frac{1}{k} \sum_{i=1}^{k} V_{k-1}(\Pi^k_{\langle i\rangle}, \theta) - V_k(\Pi^k, \theta) \Bigg] \Bigg\}$$

where $E\{\cdot \mid \theta\}$ is the expectation operator conditional on information about the quality of the firm's future products, θ' is the updated value of θ given the realized profit of the next innovation, denoted as π', and the optimization problem characterizes the firm's choice of γ. The first term on the right side is current gross profit flow accruing to the firm's product portfolio less current expenditure on R&D. The second term is the expected capital gain associated with the arrival of a new product line. Finally, because product destruction risk is equally likely across the firm's current portfolio, the last term represents the expected capital loss associated with the possibility that one among the existing product lines will be destroyed. Notice that no information about future profitability is gained or lost when a product line is destroyed.

Consider the conjecture that the solution takes the following additively separable form

$$V_k(\Pi^k, \theta) = \sum_{i=1}^{k} \frac{\pi_i}{r+\delta} + R_k(\theta) \tag{7}$$

That is, suppose that the value of the firm is the sum of the expected present value of the firm's current products plus the value of R&D activities that depends only on expectations about the profitability of future innovations and the current number of product lines. Since $V_{k+1}((\Pi^k, \pi'), \theta') = \sum_{i=1}^{k} \frac{\pi_i}{r+\delta} + \frac{\pi'}{r+\delta} + R_{k+1}(\theta')$ under the conjecture, Equation (6) implies

$$\begin{aligned} rV_k(\Pi^k, \theta) &= r\sum_{i=1}^{k} \frac{\pi_i}{r+\delta} + rR_k(\theta) \\ &= \sum_{i=1}^{k} \pi_i + k \max_{\gamma} \left\{ \gamma E\left[\frac{\pi'}{r+\delta} + R_{k+1}(\theta') - R_k(\theta) \,\middle|\, \theta \right] - wc(\gamma) \right\} \\ &\quad - \delta \left[\sum_{i=1}^{k} \frac{\pi_i}{r+\delta} + k\big[R_k(\theta) - R_{k-1}(\theta)\big] \right] \end{aligned}$$

Because the terms on the left that involve the profits of the products currently supplied cancel with those on the right, the conjecture holds for any sequence of functions $R_k(\theta)$, $k = 1, 2, \ldots$ that satisfies the functional difference equation

$$\begin{aligned} rR_k(\theta) &= k \max_{\gamma \geq 0} \left\{ \gamma E\left[\frac{\pi'}{r+\delta} + R_{k+1}(\theta') - R_k(\theta) \,\middle|\, \theta \right] - wc(\gamma) \right\} \\ &\quad - \delta k\big[R_k(\theta) - R_{k-1}(\theta)\big] \end{aligned} \tag{8}$$

In words, the return on the value of the R&D department is the expected gain in future profit associated with the next innovation plus the expected capital gains and losses to the R&D operation associated with the possibility of product creation and destruction. In general, these terms are nonzero because a new innovation changes expectations about the profitability of any future innovation and because a change in scale affects future returns to and costs of R&D.

Note that Equation (8) can be rewritten as

$$R_k(\theta) = k \max_{\gamma \geq 0} \left\{ \frac{\gamma E\left[\frac{\pi'}{r+\delta} + R_{k+1}(\theta') \,\middle|\, \theta\right] - wc(\gamma) + \delta R_{k-1}(\theta)}{r + (\delta + \gamma)k} \right\}$$

Because the right-hand side satisfies Blackwell's sufficient conditions for a contraction that maps the set of nonnegative functions defined on the product of the nonnegative reals and nonnegative integers into itself, a unique solution exists. If the uncertain profit of the next future innovation, π', is stochastically increasing in expected profitability as summarized by θ, the unique solution is an increasing function of θ for every value of k by a similar argument. Furthermore, the fact that the right-hand side is strictly increasing in k, $R_{k+1}(\theta')$, and $R_{k-1}(\theta)$ also implies that the contraction maps the functions increasing in k into itself. In sum, the solution has the properties $\theta' > \theta \Rightarrow R_k(\theta') \geq R_k(\theta)$ and $R_{k+1}(\theta) > R_k(\theta)$.

As an implication of (8), a firm's optimal product creation rate maximizes the expected net return to R&D activity,

$$\begin{aligned}(9) \qquad \gamma &= \arg\max_{\gamma} \{\gamma E[V_{k+1}((\Pi^k, \pi'), \theta') \,|\, \theta] - V_k(\Pi^k, \theta) - wc(\gamma)\} \\ &= \arg\max_{\gamma} \left\{\gamma E\left[\frac{\pi'}{r+\delta} + R_{k+1}(\theta') - R_k(\theta) \,\middle|\, \theta\right] - wc(\gamma)\right\}\end{aligned}$$

By implication, a firm's expected growth rate, the difference between the chosen creation rate γ and the market-determined destruction rate δ, is independent of the firm's current productivity and size if the profitability of the next innovation is independent of past realization of product quality. When past successes have no consequence for future prospects, there is no incentive for firms that are currently more productive to grow faster and to become larger.

4. GENERAL EQUILIBRIUM

In this section, we complete the specification of a general equilibrium model and establish that it has a solution in the special case of deterministic heterogeneity in productivity. As a corollary of the existence proof, we also find that the equilibrium is unique in the homogenous productivity case.

4.1. *Product Creation.* We restrict the analysis to the case of deterministic heterogeneity. Namely, assume that the profitability of the next innovation is π with probability 1 given that π is the profit per product of those currently supplied. Since $\theta \equiv \pi$ in this case,

$$rR_k(\pi) = k \max_{\gamma} \left\{\gamma \left(\frac{\pi}{r+\delta} + R_{k+1}(\pi) - R_k(\pi)\right) - wc(\gamma)\right\} + \delta k[R_{k-1}(\pi) - R_k(\pi)]$$

from (8), it follows that the solution for $R_k(\pi)$ is proportional to k. Namely, $R_k(\pi) = k\Delta R(\pi)$ where by substitution

$$(r+\delta)\Delta R(\pi) = \max_{\gamma \geq 0}\left\{\gamma\left(\frac{\pi}{r+\delta} + \Delta R(\pi)\right) - wc(\gamma)\right\} \tag{10}$$

is the value of R&D per product line for a firm of type π and the optimal creation rate choice is

$$\gamma(\pi) = \arg\max_{\gamma \geq 0}\left\{\gamma\left(\frac{\pi}{r+\delta} + \Delta R\right) - wc(\gamma)\right\} = \arg\max_{\gamma \geq 0}\left\{\frac{\pi - wc(\gamma)}{r+\delta-\gamma}\right\} \tag{11}$$

where the second equality is obtained by using (10) to eliminate ΔR on the left side.

From Equation (9), an interior solution for the firm's creation rate choice, denoted as $\gamma(\pi)$, satisfies the following first-order condition:

$$wc'(\gamma) = \frac{\pi}{r+\delta} + \Delta R(\pi) = \max_{\gamma \geq 0} \frac{\pi - wc(\gamma)}{r+\delta-\gamma} \tag{12}$$

Obviously, the optimal creation rate is a strictly increasing function of the firm's profit rate. We conjecture that a similar conclusion holds when expected profitability is positively correlated with past realizations, but we do not have a formal proof.

4.2. *The Distribution of Firm Size.* As the set of firms with k products at a point in time must either have had k products already and neither lost nor gained another, have had $k-1$ and innovated, or have had $k+1$ and lost one to destruction over any sufficiently short time period, the equality of the flows into and out of the set of firms of type π with $k > 1$ products requires

$$\gamma(\pi)(k-1)M_{k-1}(\pi) + \delta(k+1)M_{k+1}(\pi) = (\gamma+\delta)kM_k(\pi)$$

for every π where $M_k(\pi)$ is the steady-state mass of firms of type π that supply k products. Because an incumbent dies when its last product is destroyed by assumption, entrants flow into the set of firms with a single product at rate η,

$$\phi(\pi)\eta + 2\delta M_2(\pi) = (\gamma(\pi)+\delta)M_1(\pi)$$

where $\phi(\pi)$ is the fraction of the new entrants that realize profit π. Births must equal deaths in steady state and only firms with one product are subject to death risk. Therefore, $\phi(\pi)\eta = \delta M_1(\pi)$ and

$$M_k(\pi) = \frac{k-1}{k}\gamma(\pi)M_{k-1} = \frac{\phi(\pi)\eta}{\delta k}\left(\frac{\gamma(\pi)}{\delta}\right)^{k-1} \tag{13}$$

by induction.

The size distribution of firms conditional on type can be derived using Equation (13). Specifically, the total firm mass of type π is

$$M(\pi) = \sum_{k=1}^{\infty} M_k(\pi) = \frac{\phi(\pi)\eta}{\delta} \sum_{k=1}^{\infty} \frac{1}{k} \left(\frac{\gamma(\pi)}{\delta}\right)^{k-1}$$
$$= \frac{\eta}{\delta} \ln\left(\frac{\delta}{\delta - \gamma(\pi)}\right) \frac{\delta\phi(\pi)}{\gamma(\pi)}$$

if finite. Hence, the fraction of type π firm with k product is

$$\frac{M_k(\pi)}{M(\pi)} = \frac{\frac{1}{k}\left(\frac{\gamma(\pi)}{\delta}\right)^k}{\ln\left(\frac{\delta}{\delta - \gamma(\pi)}\right)} \tag{14}$$

This is the logarithmic distribution with parameter $0 < \gamma(\pi)/\delta < 1$, which is the ratio of the type's creation rate to the market-wide rate of destruction.[3] Consistent with the observations on firm size distributions, that implied by the model is highly skewed to the right.

The conditional mean of the distribution,

$$E\{k \mid \pi\} = \sum_{k=1}^{\infty} \frac{kM_k(\pi)}{M(\pi)} = \frac{\frac{\gamma(\pi)}{\delta - \gamma(\pi)}}{\ln\left(\frac{\delta}{\delta - \gamma(\pi)}\right)}$$

is increasing in $\gamma(\pi)$. Formally, because $(1 + a) \ln(1 + a) > a > 0$, the expected number of products produced increases with firm profitability

$$\frac{\partial E\{k \mid \pi\}}{\partial \pi} = \left(\frac{(1 + a(\pi)) \ln(1 + a(\pi)) - a(\pi)}{(1 + a(\pi)) \ln^2(1 + a(\pi))}\right) \frac{\delta\gamma'(\pi)}{(\delta - \gamma(\pi))^2} > 0 \tag{15}$$

where $a(\pi) = \gamma(\pi)/(\delta - \gamma(\pi))$ if and only if $\gamma'(\pi) > 0$.

This result is indeed consistent with the positive correlation between the value of labor productivity, $p = wq = w/(1 - \pi)$, and value added, as reflected in the number of products supplied, as observed in the Danish data. Furthermore, the relationship between the expected labor force size, which is $[(1 - \pi)/w + c(\gamma)]E\{k \mid \pi\}$, and productivity is ambiguous because the demand for labor to produce inputs of higher productivity is smaller. Hence, the model also explains the lack of correlation between labor force size and productivity reported in Table 1.

When permanent differences in product quality exist across firms, workers move from less to more profitable surviving firms as well as from exiting to entering firms.

[3] This result is in Klette and Kortum (2004). We include the derivation here simply for completeness.

This selection effect can be demonstrated by noting that more profitable firms are overrepresented (relative to their fraction at entry) among those that produce more than one product and that this "selection bias" increases with the number of products produced. Namely, the difference between the steady-state value and the value at entry of the mass of firms of a given type relative to any other,

$$\frac{M_k(\pi')}{M_k(\pi)} - \frac{\phi(\pi')}{\phi(\pi)} = \frac{\phi(\pi')}{\phi(\pi)}\left[\left(\frac{\gamma(\pi')}{\gamma(\pi)}\right)^{k-1} - 1\right] \tag{16}$$

is positive and increasing in k when for more profitable firms, i.e., when $\pi' > \pi$.

4.3. *Firm Entry and Labor Market Clearing.* The entry of a new firm requires an innovation. The cost of entry is the expected cost of the R&D effort required of a potential entrant to discover and develop a new successful product. Hence, if a potential entrant obtains ideas for new products at frequency h per period, the expected opportunity cost of her effort per innovation is w/h, the expected earnings forgone during the required period of R&D activity. As no entrant knows the profitability of its product a priori but all know its distribution by assumption, new firms enter if and only if the expected value of a new product exceeds the cost. Assuming that the condition holds, the endogenous equilibrium product destruction rate, δ, adjusts though entry to equate the expected cost and return. The equality of the expected return and the cost of entry require that

$$\sum_\pi V_1(\pi,\pi)\phi(\pi) = \sum_\pi \max_{\gamma\geq 0}\left\{\frac{\pi - wc(\gamma)}{r+\delta-\gamma}\right\}\phi(\pi) = \frac{w}{h} \tag{17}$$

from Equations (7) and (12).[4]

Because the new product arrival rate of a firm of type π with k products is $\gamma(\pi)k$ and the measure of such firms is $M_k(\pi)$ and the total mass of products is fixed, the aggregate rate of creative destruction is the sum of the entry rate and the creation rates of all the incumbents. That is,

$$\begin{aligned}\delta &= \eta + \sum_\pi\sum_{k=1}^{\infty}\gamma(\pi)kM_k(\pi) = \eta + \sum_\pi \gamma(\pi)\frac{\phi(\pi)\eta}{\delta}\sum_{k=1}^{\infty}\left(\frac{\gamma(\pi)}{\delta}\right)^{k-1}\\ &= \eta\left(\sum_\pi\phi(\pi) + \sum_\pi\frac{\gamma(\pi)\phi(\pi)}{\delta}\times\frac{\delta}{\delta-\gamma(\pi)}\right) = \eta\sum_\pi\frac{\delta\phi(\pi)}{\delta-\gamma(\pi)}\end{aligned}$$

where the second equality follows from (13) and the third is implied by the fact that $\sum_\pi \phi(\pi) = 1$ and the convergence of the infinite sum. Using the assumption that the measure of firms is unity, a direct derivation of the same relationship follows:

[4] For simplicity, we assume that the number of different product qualities is finite.

$$(18)\qquad 1=\sum_{\pi}\sum_{k=1}^{\infty}kM_k(\pi)=\sum_{\pi}\frac{\eta\phi(\pi)}{\delta}\sum_{k=1}^{\infty}\left(\frac{\gamma(\pi)}{\delta}\right)^{k-1}=\eta\sum_{\pi}\frac{\phi(\pi)}{\delta-\gamma(\pi)}$$

provided, of course, that the aggregate rate of creative destruction exceeds the creation rate of every firm type, i.e., $\delta>\gamma(\pi)$ for all π. Below, we will seek an equilibrium solution to the model that satisfies this property.

There is a fixed measure of available workers, denoted by L, seeking employment at any positive wage. In equilibrium, these are allocated across production and R&D activities, those performed by both incumbent firms and potential entrants. Since the number of workers employed for production purposes per product of quality q is $x=1/wq=(1-\pi)/w$ from Equations (4) and (5), the total number demanded for production activity by firms of type π with k products is $L_x(k,\pi)=k(1-\pi)/w>0$. The number of R&D workers employed by incumbent firms of type π with k products is $L_R(k,\pi)=kc(\gamma(\pi))$. Because a potential entrant innovates at frequency h, the total number so engaged in R&D is $L_E=\eta/h$, given entry rate η. Hence, the equilibrium wage satisfies the labor market clearing condition

$$\begin{aligned}(19)\qquad L&=\sum_{\pi}\sum_{k=1}^{\infty}[L_x(k,\pi)+L_R(k,\pi)]M_k(\pi)+L_E\\&=\sum_{\pi}\sum_{k=1}^{\infty}\left(\frac{1-\pi}{w}+c(\gamma(\pi))\right)kM_k(\pi)+\frac{\eta}{h}\\&=\sum_{\pi}\left(\frac{1-\pi}{w}+c(\gamma(\pi))\right)\frac{\phi(\pi)\eta}{\delta}\sum_{k=1}^{\infty}\left(\frac{\gamma(\pi)}{\delta}\right)^{k-1}+\frac{\eta}{h}\\&=\eta\left(\sum_{\pi}\left(\frac{1-\pi}{w}+c(\gamma(\pi))\right)\frac{\phi(\pi)}{\delta-\gamma(\pi)}+\frac{1}{h}\right)\end{aligned}$$

where again the last equality is implied by Equation (13) and the requirement that $\delta>\gamma(\pi)$ for all π.

4.4. *Existence.*

DEFINITION. A steady-state *market equilibrium* is a triple composed of a labor-market-clearing wage w, entry rate η, and creative destruction rate δ that satisfies Equations (17), (18), and (19) provided that the optimal creation rate choice of any firm type, defined by Equation (11), is less than the rate of creative destruction, i.e., $\gamma(\pi)<\delta$ for all π in the support of the type distribution at entry characterized by $\phi(\pi)$.

THEOREM. *If the cost of the R&D function*, $c(\gamma)$, *is strictly convex and* $c'(0)=c(0)=0$, *then a steady-state market equilibrium exists for all values of the labor force, L, that is sufficiently large. In the case of a single firm type, there is only one equilibrium.*

PROOF. From (17), the free-entry condition can be written as

$$\sum_{\pi} \max_{\gamma \geq 0} \left\{ \frac{\pi - wc(\gamma)}{r + \delta - \gamma} \right\} \phi(\pi) = \frac{w}{h} \tag{20}$$

By using Equation (18) to eliminate the entry rate η and Equation (20) to eliminate w/h in the labor-market-clearing condition, Equation (19), one can write the result as

$$\begin{aligned} \frac{wL}{\eta} &= wL \sum_{\pi} \frac{\delta}{\delta - \gamma(\pi)} \phi(\pi) \\ &= \delta \sum_{\pi} \left(\frac{1 - \pi + wc(\gamma(\pi))}{\delta - \gamma(\pi)} + \max_{\gamma \geq 0} \frac{\pi - wc(\gamma)}{r + \delta - \gamma} \right) \phi(\pi) \\ &= \sum_{\pi} \frac{\delta}{\delta - \gamma(\pi)} \left(1 - r \max_{\gamma \geq 0} \frac{\pi - wc(\gamma(\pi))}{r + \delta - \gamma} \right) \phi(\pi) \end{aligned}$$

where the last equality is implied by the fact that $\gamma(\pi)$ is the optimal choice of the creation rate for a type π firm and a little algebra. Hence,

$$1 = wL + \frac{r \sum_{\pi} \left(\max_{\gamma \geq 0} \frac{\pi - wc(\gamma)}{r + \delta - \gamma} \right) \frac{\phi(\pi)}{\delta - \gamma(\pi)}}{\sum_{\pi} \frac{\phi(\pi)}{\delta - \gamma(\pi)}} \tag{21}$$

Since the aggregate value added in the economy is unity by choice of the numeraire, this expression is the income identity. Namely, the total wage bill plus the return on the values of all the operating firms in the economy is equal to value added.

Given that $c(\gamma)$ is increasing and strictly convex, and that $c(0) = c'(0) = 0$, the optimal creation rate for each type conditional on the market wage and rate of creative destruction is uniquely determined by the first-order condition stated as Equation (12). Since the optimal creation rate is strictly increasing in productivity and strictly decreasing in market wage, the optimal creation rate is less than the rate of creative destruction for all π, $\gamma(\pi) < \delta\ \forall \pi \in [\underline{\pi}, \bar{\pi}]$, at any point (w, δ) above the curve defined by $\gamma(\bar{\pi}) = \delta$, which is

$$w = \frac{\bar{\pi}}{rc'(\delta) + c(\delta)} \tag{22}$$

from (12). This boundary of the admissible set of (w, δ) satisfying the convergence requirement is labeled *BB* in Figure 3. As illustrated, the wage on the boundary is positive, tends to infinity as δ tends to zero, is strictly decreasing in δ, and tends to zero as δ tends to infinity, given the assumed properties of the R&D cost function.

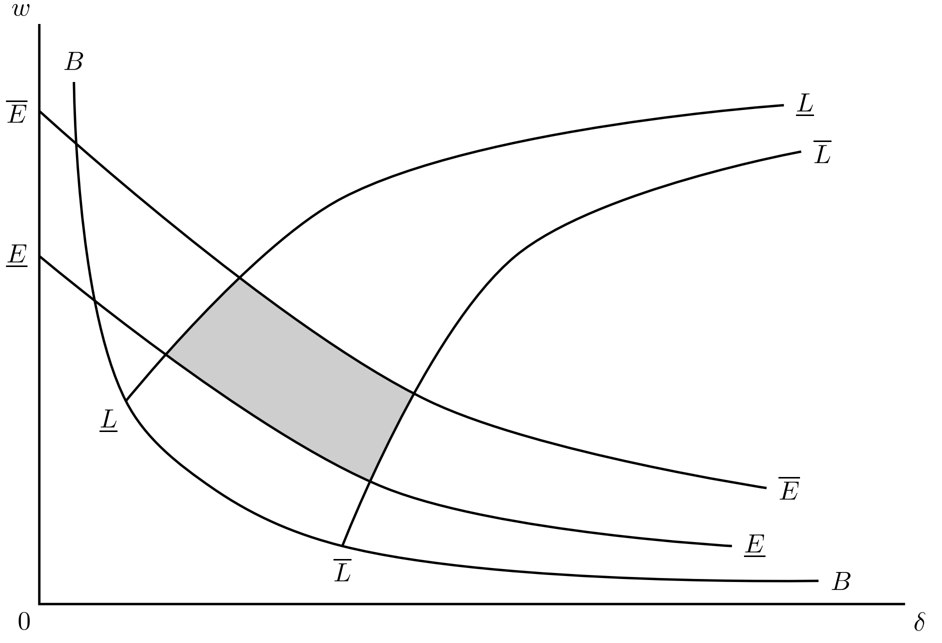


FIGURE 3

EQUILIBRIUM WAGE AND DESTRUCTION RATE

An equilibrium is any (w, δ) pair satisfying Equations (20) and (21) provided that it lies above the boundary BB. Let $w = E_\pi(\delta)$ represent the locus of points implicitly defined by

$$\max_{\gamma \geq 0} \frac{\pi - wc(\gamma)}{r + \delta - \gamma} = \frac{w}{h} \tag{23}$$

and let $w = L_\pi(\delta)$ represent solution to

$$1 = wL + r\left(\max_{\gamma \geq 0} \frac{\pi - wc(\gamma)}{r + \delta - \gamma}\right) \tag{24}$$

in the region bounded by (22). It is straightforward to show that $E'_\pi(\delta) < 0$. Because $\pi > 0$, Equation (24) implies $L > \frac{c(\gamma)}{r+\delta-\gamma}$ when $\gamma(\pi) < \delta$, it also follows that $L'_\pi(\delta) > 0$ in the region above the BB curve. Although the curve of $E_\pi(\delta)$ lies below the boundary BB at $\delta = 0$, eventually it intersects the line and remains above it for all larger values of δ as drawn in the figure. Indeed, the curve defined by $w = E_{\bar{\pi}}(\delta)$, labeled $\overline{EE}$ in the figure intersects BB at the unique solution to $hc'(\delta) = 1$.

In Figure 3, the curves $\overline{LL}$ and $\underline{LL}$ represent $w = L_{\bar{\pi}}(\delta)$ and $w = L_{\underline{\pi}}(\delta)$, respectively. Similarly, $w = E_{\bar{\pi}}(\delta)$ and $w = E_{\underline{\pi}}(\delta)$ are represented as $\overline{EE}$ and $\underline{EE}$, respectively, in Figure 3. Because

$$\max_{\gamma \geq 0}\left\{\frac{\underline{\pi} - wc(\gamma)}{r + \delta - \gamma}\right\} \leq \max_{\gamma \geq 0}\left\{\frac{\pi - wc(\gamma)}{r + \delta - \gamma}\right\} \leq \max_{\gamma \geq 0}\left\{\frac{\bar{\pi} - wc(\gamma)}{r + \delta - \gamma}\right\} \quad \text{for all} \quad \pi \in [\underline{\pi}, \bar{\pi}]$$

it follows that (23) implies $E_{\bar{\pi}}(\delta) \geq E_{\underline{\pi}}(\delta)$ and that (24) implies $L_{\bar{\pi}}(\delta) \geq L_{\underline{\pi}}(\delta)$ with strict equality holding if and only if $\bar{\pi} > \underline{\pi}$. Furthermore, the joint solution to the equilibrium conditions (20) and (21) must lie in the intersection of the two pair of curves, the shaded area in Figure 3. Given continuity of the relationships, at least one common solution exists in that region. Finally, the shaded area lies above BB in the figure for all sufficiently large values of L because the boundary BB and the $\underline{EE}$ curve are both independent of L from Equations (22) and (23) whereas the $\underline{LL}$ curve, defined by (24), shifts down in Figure 3 without limit as L increases.

Of course, the shaded area collapses to point as $\bar{\pi} - \underline{\pi} \to 0$, a fact that implies that the equilibrium is unique, given no productive heterogeneity. ■

5. REALLOCATION AND PRODUCTIVITY GROWTH

The model developed in the article implies that firm-productive heterogeneity has important implications for the sources of aggregate growth. The aggregate rate of labor productivity growth is the product of the innovation rate and the average log of the quality improvement step size from Equation (3). Formally,

$$g = \delta Eq = \eta \sum_{\pi} \ln\left(\frac{1}{1-\pi}\right)\phi(\pi) + \sum_{\pi} \gamma(\pi) \ln\left(\frac{1}{1-\pi}\right)\sum_{k=1}^{\infty} kM_k(\pi)$$

where

$$\delta = \eta + \sum_{\pi}\sum_{k=1}^{\infty} \gamma(\pi) kM_k(\pi)$$

is the rate of creative destruction.

The model developed in this article suggests the potential importance of worker reallocation from less to more profitable continuing firms as a source of productivity growth. The following decomposition,

$$g=\eta\sum_{\pi}\ln\left(\frac{1}{1-\pi}\right)\phi(\pi)+\sum_{\pi}\gamma(\pi)\ln\left(\frac{1}{1-\pi}\right)\phi(\pi) + \sum_{\pi}\gamma(\pi)\ln\left(\frac{1}{1-\pi}\right)\left[\sum_{k=1}^{\infty}kM_k(\pi)-\phi(\pi)\right] \tag{25}$$

highlights that role. The first term is the net effect of entry and exit on productivity growth. Foster et al. (2001) find that 25–30% of productivity growth can be attributed to that source. The second term is the average contribution of continuing firms if there were no firm size selection, and the last term can be regarded as a measure of the net contribution of worker reallocation to productivity growth attributable to size selection. Because Equations (13) and (18) imply that

$$\sum_{k=1}^{\infty}kM_k(\pi)=\frac{\eta\phi(\pi)}{\delta-\gamma(\pi)}=\phi(\pi)$$

if $\gamma'(\pi)\equiv 0$, the last term is zero without selection. Furthermore, because firms that grow faster supply more products, Equation (16) implies that the distribution of product lines over product profit strictly stochastically dominates the distribution at entry when $\gamma'(\pi)>0$. Finally, because $\gamma(\pi)\ln(\frac{1}{1-\pi})$ is strictly increasing in π, the contribution of reallocation is strictly positive in this case. Note that the size of the reallocation effect, which can be written as

$$\sum_{\pi}\gamma(\pi)\ln\left(\frac{1}{1-\pi}\right)\left[\sum_{k=1}^{\infty}kM_k(\pi)-\phi(\pi)\right] = \sum_{\pi}\gamma(\pi)\ln\left(\frac{1}{1-\pi}\right)\left[\frac{\eta}{\delta-\gamma(\pi)}-1\right]\phi(\pi)$$

depends on the extent of the initial dispersion in firm profitability and the sensitivity of the innovation rate with respect to firm profit.

6. CONCLUDING REMARKS

Large and persistent differences in firm productivity and size exist. Evidence suggests that the reallocation of workers across firms and establishments is an important source of aggregate economic growth. In this article, we explore a variant of the Schumpeterian model of firm size evolution developed by Klette and Kortum (2004) for insights regarding these and other empirical regularities. In our version of the model, entering firms that can develop more profitable products acquire more product lines in the future. Worker reallocation from less to more profitable firms induced by this product size selection contributes to aggregate productivity growth. Furthermore, the model is consistent with the observation that there is no correlation between employment size and labor productivity and a positive correlation between value added and labor productivity observed in

Danish firm data. Finally, we prove the existence of an equilibrium solution to the model.

In a related paper in progress (Lentz and Mortensen, 2005), we estimate the structure of the model using the same Danish firm data described here. The model's structural parameters are identified by the empirical firm size distribution observed, the patterns of firm size evolution, and the correlations between firm productivity, labor force size, and value added found in the data. With these estimates, we quantify the contribution of worker reallocation to productivity growth implied by the model as well as the optimal rate of growth.

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