Lecture 9: Back to Dynamic Programming

Economics 712, Fall 2014

1 Dynamic Programming

1.1 Constructing Solutions to the Bellman Equation

Bellman equation:

$$V(x) = \sup_{y \in \Gamma(x)} \{F(x,y) + \beta V(y)\}$$

Assume:

(Γ1): $X \subseteq \mathbb{R}^l$ is convex, $\Gamma : X \rightrightarrows X$ nonempty, compact-valued, continuous

(F1:) $F : A \to \mathbb{R}$ is bounded and continuous, $0 < \beta < 1$.

Define the **Bellman operator**:

$$(Tf)(x) = \max_{y \in \Gamma(x)} \{F(x,y) + \beta f(y)\}$$

Theorem: Under (Γ1), (F1), $T : C(X) \to C(X)$ is a contraction, and hence has a unique fixed point $v \in C(X)$ and for all $v_0 \in C(X)$, $\|T^n v_0 - v\| \leq \beta^n \|v_0 - v\|$. Moreover, the optimal policy correspondence:

$$G(x) = \{y \in \Gamma(x) : v(x) = F(x,y) + \beta v(y)\}$$

is compact-valued and uhc.

In addition, assume:

(F2): For each y, $F(\cdot, y)$ is strictly increasing

(Γ2): Γ is monotone: $x \leq x' \Rightarrow \Gamma(x) \subseteq \Gamma(x')$.

Theorem: Under (Γ1)-(Γ2), (F1)-(F2), the value function v solving (FE) is strictly increasing.

Alternatively (or in addition), assume:

(F3): F is strictly concave

(Γ3): Γ is convex

Theorem: Under (Γ1),(Γ3), (F1),(F3), the value function v solving (FE) is strictly concave, and the G is a continuous, single-valued optimal policy function.

1.2 Differentiability of the Value Function

Theorem (Benveniste-Scheinkman): Let $X \subseteq \mathbb{R}^l$ be convex, $V : X \to \mathbb{R}$ be concave. Take $x_0 \in \text{int}X$, D open neighborhood of x_0. If there exists a concave, differentiable function $W : D \to \mathbb{R}$ with $W(x_0) = V(x_0)$ and $W(x) \leq V(x) \quad \forall x \in D$, then V is differentiable at x_0 and $V_x(x_0) = W_x(x_0)$.

Assume:

(F4): $F : A \to \mathbb{R}$ is continuously differentiable on the interior of A.

Theorem: Under (Γ1),(Γ3), (F1),(F3),(F4) let v be the value function solving (FE) is strictly concave, and the g be the optimal policy function. If $x_0 \in \text{int}X$ and $g(x_0) \in \text{int}\Gamma(x_0)$ then v is continuously differentiable at x_0 with:

$$V_x(x_0) = F_x(x_0, g(x_0))$$

1.3 Euler Equations

Under the previous assumptions, first-order conditions and envelope (Benveniste-Scheinkman) characterize solution of Bellman

$$V(x) = \max_{y \in \Gamma(x)} \{F(x,y) + \beta V(y)\}$$

First order condition:

$$F_y(x, g(x)) + \beta V_x(g(x)) = 0$$

Envelope condition:

$$V_x(x) = F_x(x, g(x))$$

Combine for Euler equation (functional)

$$F_y(x, g(x)) + \beta F_x(g(x), g(g(x))) = 0$$

Can also be derived via variational argument.

1.4 Linear-Quadratic Problems

Consider minimization problem:

$$\min_{\{v_t\}} \sum_{t=0}^{\infty} \beta^t \left(y_t' Q y_t + v_t' R v_t\right)$$

subject to:

$$y_{t+1} = A y_t + B v_t$$

Transform to remove discounting: $x_t = \beta^{t/2} y_t$, $u_t = \beta^{t/2} v_t$.

$$\min_{\{u_t\}} \sum_{t=0}^{\infty} \left(x_t' Q x_t + u_t' R u_t\right)$$

subject to:

$$x_{t+1} = Ax_t + Bu_t$$

Bellman equation:

$$V(x) = \min_u \{x'Qx + u'Ru + V(Ax + Bu)\}$$

Guess and verify: $V(x) = x'Wx$.

$$x'Wx = \min_u \{x'Qx + u'Ru + (Ax + Bu)'W(Ax + Bu)\}$$

First-order condition:

$$\begin{aligned} Ru &+ B'W(Ax + Bu) = 0 \\ u &= -(R + B'WB)^{-1}B'WAx \\ &\equiv -Fx \end{aligned}$$

Substitute back in, verify that W solves the (discrete) algebraic Riccati equation:

$$W = Q + A'WA - A'WB(R + B'WB)^{-1}B'WA$$

2 Consumption Savings-Problem: Infinite Horizon

2.1 Basic Problem

$$\max_{\{c_t\}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to:

$$x_{t+1} = R(x_t - c_t + y_t)$$

Can formulate in SLP terms, noting $c_t = x_t + y_t - x_{t+1}/R$.

Conditions easy to verify if u bounded, $u' > 0, u'' < 0$.

Write choice variable as savings $s = x - c + y$:

$$V(x) = \max_s \{u(x + y - s) + \beta V(Rs)\}$$

First-order condition, $s = g(x)$:

$$u'(x + y - g(x)) = \beta R V'(Rg(x))$$

Envelope condition:

$$V'(x) = u'(x + y - g(x))$$

Combine to get (functional) consumption Euler equation:

$$u'(x + y - g(x)) = \beta R u'(Rg(x) + y - g(g(x)))$$

2.2 A Solvable Example

Set $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$, $y = 0$.

Guess $V(x) = \frac{A}{1-\gamma} x^{1-\gamma}$

$$Ax^{1-\gamma} = \max_s \left\{ \frac{(x-s)^{1-\gamma}}{1-\gamma} + \beta A (Rs)^{1-\gamma} \right\}$$

Can verify that:

$$\begin{aligned} A &= \left(1 - \beta^{\frac{1}{\gamma}} R^{\frac{1-\gamma}{\gamma}}\right)^{-\gamma} \\ s &= kx = \beta^{\frac{1}{\gamma}} R^{\frac{1-\gamma}{\gamma}} \end{aligned}$$