Lecture 8: Dynamic Programming

Economics 712, Fall 2014

1 Basic Consumption-Savings Problem

1.1 Characterization of Solution

Reconsider date $T-1$ problem:

$$V_{T-1}(x) = \max_{0 \leq c \leq x+y} \{u(c) + \beta V_T(R(x-c+y))\}$$

First order condition:

$$\begin{aligned} u'(c) &= \beta R V_T'(R(x-c+y)) \\ &= \beta R u'(R(x-c+y)+y) \end{aligned}$$

determines $c_{T-1} = c_{T-1}(x)$. Then:

$$V_{T-1}(x) = u(c_{T-1}(x)) + \beta V_T(R(x - c_{T-1}(x) + y))$$

So V_{T-1} is continuous, differentiable. Then we have the envelope condition:

$$V_{T-1}'(x) = u'(c_{T-1}(x) + y)$$

Similar conditions hold for any t:

$$\begin{aligned} u'(c_t(x)) &= \beta R u'(R(x - c_t(x) + y) + y), \quad \text{or:} \\ u'(c_t) &= \beta R u'(c_{t+1}) \end{aligned}$$

Consumption Euler equation

1.2 Permanent Income Theory

Suppose $\beta R = 1$, then $c_t = \bar{c} \forall t$. Then intertemporal constraint gives:

$$\bar{c} = \frac{\sum_{t=0}^{T} \left(\frac{1}{R}\right)^t y_t + x_0}{\sum_{t=0}^{T} \left(\frac{1}{R}\right)^t}$$

Even with large variation in $\{y_t\}$ saving and borrowing used to smooth consumption.

2 Dynamic Programming

2.1 Sequence and Recursive Formulations

(SP) Sequence problem:

$$V^*(x_0) = \sup_{\{x_{t+1}\}} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1})$$

subject to

$$x_{t+1} \in \Gamma(x_t), \quad x_0 \text{ given}$$

(FE) Bellman equation:

$$V(x) = \sup_{y \in \Gamma(x)} \{F(x, y) + \beta V(y)\}$$

Define set of feasible plans (from x_0):

$$\Pi(x_0) = \{\{x_t\} : x_{t+1} \in \Gamma(x_t) \ \ \forall t\}$$

Theorem: If $\Gamma(x)$ is nonempty for all $x \in X$ and $\forall x_0 \in X$ and $x \in \Pi(x_0)$:

$$\lim_{T \to \infty} \sum_{t=0}^{T} \beta^t F(x_t, x_{t+1}) \ \text{ exists}$$

Then if v solves (FE) and satisfies:

$$\lim_{T\to\infty} \beta^T v(x_T) = 0$$

then $v = v^*$.

2.2 Contraction Mappings

Definition: $T : S \to S$ is a contraction with modulus β on the metric space (S, μ) if $\mu(Tx, Ty) \leq \beta\mu(x, y)$ for all $x, y \in S$, for some $0 < \beta < 1$.

Theorem (Contraction Mapping): If (S, μ) is a complete metric space and $T : S \to S$ is a contraction, then T has a unique fixed point $v \in S$, $v = Tv$.

Note that for any $v_0 \in S$, $\mu(T^n, v) \leq \beta^n \mu(v_0, v)$.

Corollary: (S, μ) is a complete metric space and $T : S \to S$ is a contraction with $v = Tv$. If $S' \subseteq S$ and S' is closed, and $T(S') \subseteq S'$, then $v \in S'$. If in addition, $T(S') \subseteq S'' \subseteq S'$ then $v \in S''$.

Theorem (Blackwell's Sufficient Conditions): Let $X \subseteq \mathbb{R}^l$, $B(X)$ set of bounded functions $f : X \to \mathbb{R}$ with $\|f\| = \sup_{x\in X} |f(x)|$, so $\mu(f, g) = \|f - g\|$. Let $T : B(X) \to B(X)$ satisfy:

1. Monotonicity: For $f, g \in B(X)$, if $f(x) \leq g(x)\ \forall x \in X$ then $Tf(x) \leq Tg(x)\ \forall x \in X$.

2. Discounting: There exists a $0 < \beta < 1$ such that:

$$T(f + a)(x) \leq (Tf)(x) + \beta a$$

for $a > 0$, $f \in B(x)$, all $x \in X$.

Then T is a contraction with modulus β.

2.3 Constructing Solutions to the Bellman Equation

Assume:

(Γ1): $X \subseteq \mathbb{R}^l$ is convex, $\Gamma : X \rightrightarrows X$ nonempty, compact-valued, continuous

(F1:) $F : A \to \mathbb{R}$ is bounded and continuous, $0 < \beta < 1$.

Let $C(X)$ be the space of bounded, continuous functions on X with $\|f\| = \sup_{x \in X} |f(x)|$. Note that $(C(X), \|\cdot\|)$ is a Banach space (complete normed vector space).

Define the **Bellman operator**:

$$(Tf)(x) = \max_{y \in \Gamma(x)} \{F(x, y) + \beta f(y)\}$$

Theorem: Under (Γ1), (F1), $T : C(X) \to C(X)$ is a contraction, and hence has a unique fixed point $v \in C(X)$ and for all $v_0 \in C(X)$:

$$\|T^n v_0 - v\| \leq \beta^n \|v_0 - v\|$$

Moreover, the optimal policy correspondence:

$$G(x) = \{y \in \Gamma(x) : v(x) = F(x, y) + \beta v(y)\}$$

is compact-valued and uhc.

In addition, assume:

(F2): For each y, $F(\cdot, y)$ is strictly increasing

(Γ2): Γ is monotone: $x \leq x' \Rightarrow \Gamma(x) \subseteq \Gamma(x')$.

Theorem: Under (Γ1)-(Γ2), (F1)-(F2), the value function v solving (FE) is strictly increasing.

Alternatively (or in addition), assume:

(F3): F is strictly concave

(Γ3): Γ is convex

Theorem: Under (Γ1),(Γ3), (F1),(F3), the value function v solving (FE) is strictly concave, and the G is a continuous, single-valued optimal policy function.

2.4 Differentiability of the Value Function

Theorem (Benveniste-Scheinkman): Let $X \subseteq \mathbb{R}^l$ be convex, $V : X \to \mathbb{R}$ be concave. Take $x_0 \in \text{int}X$, D open neighborhood of x_0. If there exists a concave, differentiable function $W : D \to \mathbb{R}$ with $W(x_0) = V(x_0)$ and $W(x) \leq V(x) \quad \forall x \in D$, then V is differentiable at x_0 and $V_x(x_0) = W_x(x_0)$.

Assume:

(F4): $F : A \to \mathbb{R}$ is continuously differentiable on the interior of A.

Theorem: Under (Γ1),(Γ3), (F1),(F3),(F4) let v be the value function solving (FE) is strictly concave, and the g be the optimal policy function. If $x_0 \in \text{int}X$ and $g(x_0) \in \text{int}\Gamma(x_0)$ then v is continuously differentiable at x_0 with:

$$V_x(x_0) = F_x(x_0, g(x_0))$$