Lecture 7: Consumption-Savings Problem, Finite Horizon

Economics 712, Fall 2014

1 Basic Consumption-Savings Problem

1.1 Setup

Agent preferences:

$$\sum_{t=0}^{T} \beta^t u(c_t)$$

Where u is C^2, $u' > 0$, $u'' < 0$.

Sometimes we'll also assume u is bounded, satisfies Inada: $\lim_{c\to 0} u'(c) = \infty$.

Initial wealth x_0, constant gross return $R > 1$, exogenous labor income $\{y_t\}$. So flow constraint:

$$x_{t+1} = R(x_t - c_t + y_t)$$

Intertermporal constraint:

$$\sum_{t=0}^{T} \left(\frac{1}{R}\right)^t c_t = \sum_{t=0}^{T} \left(\frac{1}{R}\right)^t y_t + x_0$$

1.2 Sequence Problem

$$\max_{\{c_t\}} \sum_{t=0}^{T} \beta^t u(c_t)$$

subject to:

$$\sum_{t=0}^{T} \left(\frac{1}{R}\right)^t c_t = \sum_{t=0}^{T} \left(\frac{1}{R}\right)^t y_t + x_0$$

Single optimization problem over consumption plan.

Standard arguments show that a solution exists and is unique.

1.3 Theorem of the Maximum

For general optimization problem, with $X \subseteq \mathbb{R}^l$, $Y \subseteq \mathbb{R}^m$.

Objective function $f : X \times Y \to \mathbb{R}$ continuous

Feasible correspondence $\Gamma : X \rightrightarrows Y$ compact valued, continuous (upper- and lower-hemicontinuous)

Maximization problem and value:

$$h(x) = \max_{y \in \Gamma(x)} f(x, y)$$

Set of maximizers:

$$G(x) = \{y \in \Gamma(x) : f(x, y) = h(x)\}$$

Theorem: Under these conditions, h is continuous and G is nonempty, compact-valued, and upper-hemicontinuous.

1.4 Recursive Formulation

Since finite horizon, solve by backward induction. For simplicity assume $y_t \equiv y \ \forall t$.

At date T, $c_T = x_T + y$. Define $V_T(x) = u(x + y)$.

At date $T-1$ solve 2 period problem, with $x = x_{T-1}$, $y = y_{T-1}$:

$$\begin{aligned} V_{T-1}(x) &= \max_{0 \le c \le x+y} \{u(c) + \beta u(R(x-c+y)+y)\} \\ &= \max_{0 \le c \le x+y} \{u(c) + \beta V_T(R(x-c+y))\} \end{aligned}$$

by the Theorem of the Maximum, $V_{T-1}(x)$ is continuous.

Continuing in this way, for any $0 \le t \le T-1$ we have:

$$V_t(x) = \max_{0 \le c \le x+y} \{u(c) + \beta V_{t+1}(R(x-c+y))\}$$

which is a (time-dependent) Bellman equation.

1.5 Principle of Optimality (Finite Horizon)

We can justify the Bellman equation with a simple case of the principle of optimality.

Let $\{c_t^*\}_{t=0}^T$ solve the sequence problem with initial wealth x_0. Given arbitrary dates $0 \le a < b \le T-1$, let x_a^* and x_{b+1}^* be the optimal wealth. Then consider the problem:

$$\max_{\{c_t\}_{t=a}^b} \sum_{t=a}^{b} \beta^{t-a} u(c_t)$$

subject to:

$$x_{t+1} = R(x_t - c_t + y), \quad x_a^*, x_{b+1}^* \text{ given}$$

Then the solution to this problem is $\{c_t^*\}_{t=a}^b$.

Bellman equation is a special case.

1.6 Characterization of Solution

Reconsider date $T-1$ problem:

$$V_{T-1}(x) = \max_{0 \le c \le x+y} \{u(c) + \beta V_T(R(x-c+y))\}$$

First order condition:

$$\begin{aligned} u'(c) &= \beta R V_T'(R(x-c+y)) \\ &= \beta R u'(R(x-c+y)+y) \end{aligned}$$

determines $c_{T-1} = c_{T-1}(x)$. Then:

$$V_{T-1}(x) = u(c_{T-1}(x)) + \beta V_T(R(x - c_{T-1}(x) + y))$$

So V_{T-1} is continuous, differentiable. Then we have the envelope condition:

$$V_{T-1}'(x) = u'(c_{T-1}(x) + y)$$

Similar conditions hold for any t:

$$\begin{aligned} u'(c_t(x)) &= \beta R u'(R(x - c_t(x) + y) + y), \quad \text{or:} \\ u'(c_t) &= \beta R u'(c_{t+1}) \end{aligned}$$

Consumption Euler equation

1.7 Permanent Income Theory

Suppose $\beta R = 1$, then $c_t = \bar{c} \forall t$. Then intertemporal constraint gives:

$$\bar{c} = \frac{\sum_{t=0}^{T} \left(\frac{1}{R}\right)^t y_t + x_0}{\sum_{t=0}^{T} \left(\frac{1}{R}\right)^t}$$

Even with large variation in $\{y_t\}$ saving and borrowing used to smooth consumption.