1 Basic Consumption-Savings Problem

1.1 Setup

Agent preferences:
\[ \sum_{t=0}^{T} \beta^t u(c_t) \]

Where \( u \) is \( C^2 \), \( u' > 0 \), \( u'' < 0 \).

Sometimes we’ll also assume \( u \) is bounded, satisfies Inada: \( \lim_{c \to 0} u'(c) = \infty \).

Initial wealth \( x_0 \), constant gross return \( R > 1 \), exogenous labor income \( \{y_t\} \). So flow constraint:
\[ x_{t+1} = R(x_t - c_t + y_t) \]

Intertermporal constraint:
\[ \sum_{t=0}^{T} \left( \frac{1}{R} \right)^t c_t = \sum_{t=0}^{T} \left( \frac{1}{R} \right)^t y_t + x_0 \]

1.2 Sequence Problem

\[ \max_{\{c_t\}} \sum_{t=0}^{T} \beta^t u(c_t) \]
subject to:

\[ \sum_{t=0}^{T} \left( \frac{1}{R} \right)^t c_t = \sum_{t=0}^{T} \left( \frac{1}{R} \right)^t y_t + x_0 \]

Single optimization problem over consumption plan.

Standard arguments show that a solution exists and is unique.

1.3 Theorem of the Maximum

For general optimization problem, with \( X \subseteq \mathbb{R}^l, Y \subseteq \mathbb{R}^m \).

Objective function \( f : X \times Y \rightarrow \mathbb{R} \) continuous

Feasible correspondence \( \Gamma : X \Rightarrow Y \) compact valued, continuous (upper- and lower-hemicontinuous)

Maximization problem and value:

\[ h(x) = \max_{y \in \Gamma(x)} f(x, y) \]

Set of maximizers:

\[ G(x) = \{ y \in \Gamma(x) : f(x, y) = h(x) \} \]

**Theorem:** Under these conditions, \( h \) is continuous and \( G \) is nonempty, compact-valued, and upper-hemicontinuous.

1.4 Recursive Formulation

Since finite horizon, solve by backward induction. For simplicity assume \( y_t \equiv y \ \forall t \).

At date \( T \), \( c_T = x_T + y \). Define \( V_T(x) = u(x + y) \).
At date $T - 1$ solve 2 period problem, with $x = x_{T-1}$, $y = y_{T-1}$:

$$V_{T-1}(x) = \max_{0 \leq c \leq x+y} \{ u(c) + \beta u(R(x-c+y) + y) \}$$

$$= \max_{0 \leq c \leq x+y} \{ u(c) + \beta V_T(R(x-c+y)) \}$$

by the Theorem of the Maximum, $V_{T-1}(x)$ is continuous.

Continuing in this way, for any $0 \leq t \leq T - 1$ we have:

$$V_t(x) = \max_{0 \leq c \leq x+y} \{ u(c) + \beta V_{t+1}(R(x-c+y)) \}$$

which is a (time-dependent) Bellman equation.

### 1.5 Principle of Optimality (Finite Horizon)

We can justify the Bellman equation with a simple case of the principle of optimality.

Let $\{c^*_t\}_{t=0}^T$ solve the sequence problem with initial wealth $x_0$. Given arbitrary dates $0 \leq a < b \leq T - 1$, let $x^*_a$ and $x^*_{b+1}$ be the optimal wealth. Then consider the problem:

$$\max_{\{c_t\}_{t=a}^b} \sum_{t=a}^b \beta^{t-a} u(c_t)$$

subject to:

$$x_{t+1} = R(x_t - c_t + y), \quad x_a^*, x_{b+1}^* \text{ given}$$

Then the solution to this problem is $\{c^*_t\}_{t=a}^b$.

Bellman equation is a special case.
1.6 Characterization of Solution

Reconsider date $T - 1$ problem:

$$V_{T-1}(x) = \max_{0 \leq c \leq x+y} \{u(c) + \beta V_T(R(x-c+y))\}$$

First order condition:

$$u'(c) = \beta RV_T'(R(x-c+y))$$

$$= \beta Ru'(R(x-c+y) + y)$$

determines $c_{T-1} = c_{T-1}(x)$. Then:

$$V_{T-1}(x) = u(c_{T-1}(x)) + \beta V_T(R(x-c_{T-1}(x) + y))$$

So $V_{T-1}$ is continuous, differentiable. Then we have the envelope condition:

$$V'_{T-1}(x) = u'(c_{T-1}(x) + y)$$

Similar conditions hold for any $t$:

$$u'(c_t(x)) = \beta Ru'(R(x-c_t(x) + y) + y), \text{ or:}$$

$$u'(c_t) = \beta Ru'(c_{t+1})$$

Consumption Euler equation

1.7 Permanent Income Theory

Suppose $\beta R = 1$, then $c_t = \bar{c} \forall t$. Then intertemporal constraint gives:

$$\bar{c} = \frac{\sum_{t=0}^{T} \left(\frac{1}{R}\right)^t y_t + x_0}{\sum_{t=0}^{T} \left(\frac{1}{R}\right)^t}$$

Even with large variation in $\{y_t\}$ saving and borrowing used to smooth consumption.