Lecture 19
Search Theoretic Models of Money

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Economics 712
Essential Models of Money

• Hahn (1965): money is essential if it allows agents to achieve allocations they cannot achieve with other mechanisms that also respect the enforcement and information constraints in the environment.

• Why do we care about essential models of money?

• Three frictions that will make money essential:
  1. Double-coincidence of wants problem.
  2. Long-run commitment cannot be enforced.
  3. Agents are anonymous: histories are not public information.

• Money is a consequence of these frictions in trade: medium of exchange.
Three Generations of Models

1. 1 unit of money, 1 unit of good: Kiyotaki and Wright (1993).

2. 1 unit of money, endogenous units of good: Trejos and Wright (1995).

3. Endogenous units of money, endogenous units of good: Lagos and Wright (2005).
Environment

- [0, 1] continuum of anonymous agents.

- Live forever and discount future at rate $r$.

- [0, 1] continuum of goods. Good $i$ is produced by agent $i$.

- Goods are non-storable: no commodity money.

- Unit cost of production $c \geq 0$. 
Double-Coincidence of Wants Problem

- I do not produce what I like (non-restrictive: home production, specialization).

- $iW_j$: agent $i$ likes to consume good produced by agent $j$:
  1. utility $u > c$ from consuming $j$.
  2. utility 0 otherwise.

- Probabilities of matching:

  $$ p(iWi) = 0 $$
  $$ p(jWi) = x $$
  $$ p(jWi|iWj) = y $$
First Generation: Fixed Money and Fixed Good

- Exogenously given quantity $M \in [0, 1]$ of an indivisible unit of storable good.

- Holding money yields zero utility $\gamma$: fiat money.

- Initial endowment: $M$ agents are randomly endowed with one unit of money.

- Agents holding money cannot produce (for example because you need to consume before you can produce again).

- We eliminate (non-trivial) distributions.
Trades

• Pairwise random matching of agents with Poisson arrival time $\alpha$.

• Bilateral trading is important, randomness is not (Corbae, Temzelides, Wright, 2003).

• Upon meeting, agents decide whether to trade. Then, they part company and re-enter the process.

• History of previous trades is unknown.

• Exchange 1 unit of good for 1 unit of good (barter) or 1 unit of money.
Individual Trading Strategies

- Agents never accept a good in trade if he does not like to consume it since it is not storable.

- They will barter if they like the both agents in the pair like each other goods.

- Would they accept money for goods and viceversa?

- We will look at stationary and symmetric Nash equilibria.
Probabilities

- You meet someone with arrival rate $\alpha$.

- This person can produce with probability $1 - M$.

- With probability $x$ you like what he produces.

- With probability $\pi = \pi_0 \pi_1$ (endogenous objects to be determined) both of you want to trade.

- If $\pi > 0$, we say that money circulates.
Value Functions

- Value functions with money, $V_1$:
  \[ rV_1 = \alpha x (1 - M) \pi (u + V_0 - V_1) \]

- Value functions without money, $V_0$.
  \[ rV_0 = \alpha xy (1 - M)(u - c) + \alpha x M \pi (V_1 - V_0 - c) \]

- Renormalize $\alpha x = 1$ by picking time units:
  \[ rV_1 = (1 - M) \pi (u + V_0 - V_1) \]
  \[ rV_0 = y (1 - M)(u - c) + M \pi (V_1 - V_0 - c) \]
Individual Trading Strategies

- Net gain from trading goods for money:
  \[
  \Delta_0 = V_1 - V_0 - c = \frac{(1 - M)(\pi - y)(u - c) - rc}{r + \pi}
  \]

- Net gain from trading money from goods:
  \[
  \Delta_1 = u + V_0 - V_1 = \frac{(M\pi + (1 - M)y)(u - c) + ru}{r + \pi}
  \]
Equilibrium Conditions for $\pi_0$ and $\pi_1$

- Clearly:

\[
\pi_j \begin{cases} 
  = 1 & \text{as } \Delta_j > 0 \\
  \in [0, 1] & \text{as } \Delta_j = 0 \\
  = 0 & \text{as } \Delta_j < 0 
\end{cases}
\]

- Plug those into the individual trading strategies, and check them.
Characterizing $\pi$

- Clearly $\Delta_1 > 0$. Hence $\pi_1 = 1$, i.e., the agent with money always wants to trade.

- For $\pi_0$, you have

  $$\Delta_0 = \frac{(1 - M)(u - c)\pi_0}{r + \pi_0} - \frac{(1 - M)y(u - c) + rc}{r + \pi_0}$$

- Then, $\Delta_0$ has the same sign as

  $$\pi_0 - \frac{rc + (1 - M)y(u - c)}{(1 - M)(u - c)} = \pi_0 - \hat{\pi}$$
Multiple Equilibria

- Nonmonetary equilibrium: we have an equilibrium where $\pi_0 = 0$.

- Monetary equilibrium: if
  \[
  c < \frac{(1 - M)(1 - y)}{r + (1 - M)(1 - y)} u
  \]
  then $\hat{\pi} < 1$ and $\pi_0 = 1$ is an equilibrium as well.

- Mixed-monetary equilibrium: $\pi_0 = \pi^\wedge$. However, not robust (Schevchenko and Wright, 2004).
Equilibria in \((\gamma, r)\)-Space When Money Holders Cannot Produce

\[
\begin{align*}
\pi_0 &= 0 \\
\pi_1 &= 1 \\
\end{align*}
\]

\[
\begin{align*}
\pi_0 &= 0, \\
\pi_0 &= 1 \quad \text{or} \\
\pi_0 &\in (0, 1) \\
\pi_1 &= 1 \\
\end{align*}
\]

\[
\begin{align*}
\pi_0 &= 1 \\
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\end{align*}
\]

\[
\begin{align*}
\pi_0 &= 1 \\
\pi_1 &= 0 \\
\end{align*}
\]
Welfare

- Define welfare as the average utility:
  \[ W = MV_1 + (1 - M)V_0 \]

- Then:
  \[ rW = (1 - M) \left[ (1 - M)y + M\pi \right] (u - c) \]

- Note that welfare is increasing in \( \pi \).
Welfare $\pi = 1$

- Note:

$$rW = (1 - M) [ (1 - M)y + M ] (u - c)$$

- Maximize $W$ with respect to $M$:

$$M^* = \begin{cases} 
\frac{1 - 2y}{2 - 2y} & \text{if } y < \frac{1}{2} \\
0 & \text{if } y \geq \frac{1}{2}
\end{cases}$$

- Intuition: facilitate trade versus crowding out barter.
Welfare $\pi = 0$

- Note:

$$rW = (1 - M) [(1 - M) y] (u - c)$$

- Monotonically decreasing in $M \Rightarrow M^* = 0$.

- Result is a little bit silly: it depends on the absence of free disposal of money. Otherwise, welfare is independent of $M$. 
Welfare $\pi$

Define $M$ such that $\pi=1$,

- Note:

$$rW = (1 - M) [(1 - M) y + M \pi] (u - c)$$

- Monotonically increasing in $M$ in the $[0, M]$ interval.
Welfare as a Function of $y$ (optimal $M$)
Comparison with Alternative Arrangements

• Imagine that we have the credit arrangement: “produce for anyone you meet that wants your good.”

• Value function

\[ rV_c = u - c \]

• Clearly

\[ rV_c > rW \]

• However, this arrangement is not self-enforceable: histories are not observed.
Second Generation: Endogenous Prices

- We make the very strong assumption that we exchanged one good for one unit of money.


- We set $y = 0$ and we let goods be divisible.

- When agents meet, they bargain about how much $q$ will be exchanged, or equivalently, about price $1/q$. 
Utility and Cost Functions

• Utility is $u(q)$ and cost of production is $c(q)$.

• Assumptions:

$$
\begin{align*}
  u(0) &= c(0) = 0 \\
  u'(0) &> c'(0) \\
  u'(0) &> 0, u''(0) \leq 0 \\
  c'(0) &> 0, c''(0) \geq 0
\end{align*}
$$

• Also, $\hat{q}$ and $q^*$ are such that

$$
\begin{align*}
  u(\hat{q}) &= c(\hat{q}) \\
  u'(\hat{q}) &= c'(\hat{q}) \\
  u'(q^*) &= c'(q^*)
\end{align*}
$$
Value Functions and Bargaining

• Take $q = Q$ as given. Then:

$$
rv_1 = (1 - M) [u (Q) + V_0 - V_1]
$$

$$
rv_0 = M [V_1 - V_0 - c (Q)]
$$

• Bargaining is the generalized Nash bargaining solution:

$$
q = \text{argmax} \left[ u(q) + \left( V_0 (Q) - T_1 \right)^\theta \times \left( V_1 (Q) - c(q) - T_0 \right)^\theta \right]
$$

$$
u(q) + V_0 \geq V_1
$$

$$V_1 - c(q) \geq V_0
$$

where $T_j$ is the threat point of the agent with $j$ units of money.

• We will set $T_j = 0$ and $\theta = 1/2$. 
Equilibria

• Necessary condition taking $V_0(Q)$ and $V_1(Q)$ as given:

$$[V_1(Q) - c(q)]u'(q) = [u(q) + V_0(Q)]c'(q)$$

• The bargaining solution defines a function

$$q = e(Q)$$

and we look at its fixed points.

• Two fixed points:

1. $q = 0$: nonmonetary equilibrium.

2. $q = q^e > 0$: monetary equilibrium.
Monetary Equilibrium in the Divisible-Goods Model

Diagram showing the relationship between $q$ and $Q$ with curves $f(Q)$, $e(Q)$, and $g(Q)$.

Key points:
- $q^e$, $q^*$, and $q_1$ as reference points.
Efficiency

• Note that the efficient outcome is $q^*$, i.e. $u'(q^*) = c'(q^*)$.

• In the monetary equilibrium:

$$u'(q^e) = \frac{u(q^e) + V_0(q^e)}{V_1(q^e) - c(q^e)}c'(q^e) > u'(q^*)$$

since $u(q^e) + V_0(q^e) > V_1(q^e) - c(q^e)$.

• Hence $q^* > q^e$, or equivalently, the price is too high.
Third Generation: Endogenous Prices and Goods

- Relax the assumption that agents hold 0 or 1 units of money.

- Problem: endogenous distribution of money that we (and the agents!) need to keep track of.


- Theoretical: