# Lecture 16: Stochastic Growth Model

## Economics 712, Fall 2014

# 1 Stochastic Growth/Real Business Cycle Model

## 1.1 Recursive Competitive Equilibrium

Firm problem:

$$\max_{K_t, N_t}[F(K_t, A_t N_t) - r_t K_t - w_t N_t]$$

Note firm problem is static, gives standard optimality conditions:

$$\begin{aligned} F_K(K, AN) &= r \\ AF_N(K, AN) &= w \end{aligned}$$

Can eliminate $N$ and write these as $w = w(K, A)$, $r = r(K, A)$.

Household: individual capital $k$, aggregate $K$. Takes as given pricing functions, law of motion for aggregate $K$.

Bellman equation:

$$V(k, K, A) = \max_{c,n,k'} \{u(c, 1-n) + \beta E[V(k', K', A)|k, A, K]\}$$

subject to

$$\begin{aligned} c + k' - (1-\delta)k &= w(K, A)n + r(K, A)k \\ \log A' &= \rho \log A + \varepsilon' \\ K &= G(K, A) \end{aligned}$$

Solution determines household policies $k' = g(k, K, A)$, $c = c(k, K, A)$, $n = n(k, K, A)$.

**Definition:** A **recursive competitive equilibrium** is a value function $V(k, K, A)$ for the household, aggregate decision rules $C(K, A), N(K, A), K' = G(K, A)$ and price functions $r(K, A)$, $w(K, A)$ such that

(i) $V$ solves the household problem, with decision rules $g, c, n$

(ii) The price functions $r$ and $w$ are consistent with the firm problem

(iii) individual and aggregate behavior are consistent: $g(K, K, A) = G(K, A)$, $c(K, K, A) = C(K, A)$, $n(K, K, A) = N(K, A)$

(iv) Aggregate feasibility is satisfied:

$$C(K, A) + G(K, A) - (1 - \delta)K = F(K, AN(K, A))$$

## 1.2 Characterization

$$V(k, K, A) = \max_{n,k'} \{u(w(K, A)n + r(K, A)k - k' + (1 - \delta)k, 1 - n) + \beta E[V(k', K', A)|k, A, K]\}$$

First order and envelope conditions:

$$\begin{aligned} u_c(c, 1 - n) &= \beta E[V_k(k', K', A)|k, A, K] \\ u_c(c, 1 - n)w(K, A) &= u_n(c, 1 - n) \\ V_k(k, K, A) &= u_c(c, 1 - n)[r(K, A) + 1 - \delta] \end{aligned}$$

Combining gives household Euler equation:

$$u_c(c, 1 - n) = \beta E[u_c(c', 1 - n')[r(K', A') + 1 - \delta]|k, A, K]$$

Impose equilibrium:

$$u_c(C, 1 - N) = \beta E\left[u_c(C', 1 - N')[F_K(K', A'N') + 1 - \delta]|k, A, K\right]$$

Also have intra-temporal optimality condition, in equilibrium:

$$\frac{u_n(C, 1-N)}{u_c(C, 1-N)} = AF_N(K, AN)$$

### 1.3 Optimal Growth

Since equilibrium is Pareto optimal, could have directly solved planner's problem:

$$V^*(K, A) = \max_{C,N,k'} \{u(C, 1-N) + \beta E[V^*(K', A)|A, K]\}$$

subject to

$$\begin{aligned} C + K' - (1-\delta)K &= F(K, AN) \\ \log A' &= \rho \log A + \varepsilon' \end{aligned}$$

Yields same intra-temporal optimality condition, Euler equation

Note $V(K, K, A) = V^*(K, A)$.

Benefit of RCE: holds even when equilibrium not optimal