Lecture 12: More on the Stochastic Consumption-Savings Problem, General Equilibrium

Economics 714, Fall 2014

1 Consumption-Savings Problem under Uncertainty

1.1 Euler Inequality

$$v(a,y) = \max_{(c,a')\in\Gamma(a,y)} \left\{ u(c) + \beta \int v(a',y')Q(y,dy') \right\}$$

If constraint binds:

$$v(a,y) = u(Ra + y - \underline{a}) + \beta \int v(\underline{a},y')Q(y,dy')$$

So then:

$$v_a(a,y) = Ru'(Ra + y - \underline{a})$$

With binding constraint (multiplier $\mu > 0$), first order condition becomes:

$$\begin{aligned} u'(c) &= \beta \int v_a(a',y')Q(y,dy') + \mu \\ &= \beta R \int u'(R\underline{a} + y' - a'(\underline{a},y))Q(y,dy') + \mu \\ &> \beta R \int u'(c') \end{aligned}$$

So in general:

$$u'(c_t) \geq \beta R E_t u'(c_{t+1})$$

with equality if $a_{t+1} > \underline{a}$

1.2 Dynamics of Consumption

Define $\theta_t = \beta^t R^t u'(c_t) \geq 0$. Euler inequality: $\theta_t \geq E_t \theta_{t+1}$.

Theorem (Martingale Convergence) Let $\{X_t\}$ be a submartingale. If $K = \sup_t E(|X_t|) < \infty$, then $X_t \to x$ with probability 1, where X is a random variable that satisfies $E(|X|) \leq K$.

Here $-\theta_t$ is a submartingale, and $\sup_t E(|-\theta_t|) = E(u'(c_0)) < \infty$. So $\theta_t \to \overline{\theta}$.

Implies if $\beta R > 1$ then since $\lim_{t\to\infty} \beta^t R^t = +\infty$ then $\lim_{t\to\infty} u'(c_t) = 0$ with probability 1, so $c_t \to \infty$.

Same conclusions hold with more delicate argument if $\beta R = 1$.

2 General Equilibrium: Endowment Economies

2.1 Lucas (1978) Asset Pricing Model

Large number of identical agents, single nonstorable consumption good (fruit), given off by productive units (trees) with net supply of 1.

Preferences satisfy usual conditions

Owner of tree receives stochastic dividend s_t with transition function $Q(s, ds')$.

Representative agent problem:

$$\max_{\{c_t, a_{t+1}\}} E_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to:

$$c_t + p_t a_{t+1} = (p_t + s_t) a_t$$

Conjecture pricing function $p_t = p(s_t)$. Then can write Bellman equation:

$$v(a, s) = \max_{(c, a')} \left\{ u(c) + \beta \int v(a', s') Q(s, ds') \right\}$$

subject to:

$$c + p(s) a' \leq (p(s) + s) a, \quad c \geq 0, \quad 0 \leq a' \leq 1$$

2.2 Equilibrium

Definition A *recursive competitive equilibrium* is a continuous function $p(s)$ and a continuous, bounded function $v(a, s)$ such that:

1. $v(a, s)$ solves the Bellman equation

2. $\forall s, \;\; v(1, s)$ is attained by $c = s$, $a' = 1$.

To characterize, note that wealth on hand is what really matters $(p(s) + s)a$. Re-write Bellman:

$$v((p(s) + s)a) = \max_{(a')} \left\{ u((p(s) + s)a - p(s)a') + \beta \int v((p(s') + s')a') Q(s, ds') \right\}$$

First order condition:

$$-u'(c(s))p(s) + \beta \int v'((p(s') + s')a')[p(s') + s'] Q(s, ds') = 0$$

Envelope condition:

$$v'((p(s) + s)a) = u'(c(s))$$

Combine to get Euler equation:

$$u'(c(s)) = \beta \int u'(c(s'))\frac{p(s') + s'}{p(s)}Q(s, ds')$$

Or, if $p_t = p(s_t)$, $R_{t+1} = \frac{p_{t+1}+s_{t+1}}{p_t}$:

$$u'(c_t) = \beta E_t[u'(c_{t+1})R_{t+1}]$$

But now this equation determines equilibrium pricing function. In equilibrium $a = a' = 1$, $c(s) = s$:

$$p(s) = \beta \int \frac{u'(s')(p(s') + s')}{u'(s)}Q(s, ds')$$