# Lecture 11: Consumption-Savings Problem Under Uncertainty

**Economics 714, Fall 2014**

# 1 Consumption-Savings Problem under Uncertainty

## 1.1 Basic Problem

$$\max_{\{c_t, a_{t+1}\}} E_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$

$$\text{s.t.} \quad c_t + a_{t+1} = Ra_t + y_t$$

$a_0, y_0$ given. Slight timing change from previously.

Constraints: $c_t \geq 0, \quad a_t \geq \underline{\mathrm{a}}$. Debt limit.

Income $y$ stochastic: $y \in Y \subseteq \mathbb{R}_+$ compact, Borel $\sigma$-algebra $\mathcal{Y}$.

$y$ follows Markov process w/transition function $Q$ on $(Y, \mathcal{Y})$.

Assume $Q$ has **Feller property:** for $f : Y \to \mathbb{R}$ bounded, continuous then:

$$E[f(y')|y] = \int f(y')Q(y, dy')$$

is bounded and continuous

State space: assets $a \in A = [\underline{\mathrm{a}}, \infty)$, Borel $\sigma$-algebra $\mathcal{A}$.

Joint state space $(a, y) \in X = A \times Y$ with Borel $\sigma$-algebra $\mathcal{X}$

Feasible correspondence:

$$\Gamma(x) = \{(c, a') : c + a' \leq Ra + y, \quad c \geq 0, \quad a' \geq \underline{\mathrm{a}}\}$$

## 1.2 Sequence Problem

At each date $c_t : Y^t \to \mathbb{R}_+$, measurable (w.r.t $\mathcal{Y}^t$).

$a_{t+1} : Y^t \to \mathbb{R}_+$, measurable (w.r.t $\mathcal{Y}^t$)

$$\begin{aligned}
v^*(a_0, y_0) &= \sup_{\{c_t(Y^t), a_{t+1}(Y^t)\}} E_0 \sum_{t=0}^{\infty} \beta^t u(c_t)] \\
&= \sup_{\{c_t(Y^t), a_{t+1}(Y^t)\}} \int \sum_{t=0}^{\infty} \beta^t u(c_t(Y^t)) Q(y_t, dy_{t+1}) Q(y_{t-1}, dy_t) \cdots Q(y_0, dy_1)
\end{aligned}$$

## 1.3 Bellman Equation

$$v(a, y) = \max_{(c,a') \in \Gamma(a,y)} \left\{ u(c) + \beta \int v(a', y') Q(y, dy') \right\}$$

Extensions of the previous results apply, principle of optimality is direct.

Define Bellman operator as before:

$$Tf(a, y) = \max_{(c,a') \in \Gamma(a,y)} \left\{ u(c) + \beta \int f(a', y') Q(y, dy') \right\}$$

**Theorem:** Under the assumptions here $T : C(X) \to C(X)$ is a contraction, and hence has a unique fixed point $v \in C(X)$ and for all $v_0 \in C(X)$:

$$\|T^n v_0 - v\| \leq \beta^n \|v_0 - v\|$$

Moreover, the optimal policy correspondence:

$$G(x) = \{(c, a') \in \Gamma(a, y) : v(a, y) = u(c) + \beta \int v(a', y') Q(y, dy')\}$$

is compact-valued and uhc.

In addition, under our standing assumptions we have the stronger results.

**Theorem:** (i) $v(a, y)$ is strictly increasing in $a$

(ii) $v(a, y)$ is strictly concave in $a$ and the optimal policy functions $c(a, y)$ and $a'(a, y)$ are continuous

(iii) If $(a_0, y_0) \in \mathrm{int(X)}$ and $(c(a_0, y_0), a'(a_0, y_0)) \in \mathrm{int}\Gamma(\mathrm{a_0, y_0})$ then $v$ is continuously differentiable (in $a$) at $(a_0, y_0)$ and:

$$v_a(a_0, y_0) = Ru'(c(a_0, y_0)) = Ru'(Ra + y - a'(a_0, y_0))$$

## 1.4 Euler Equations

If the constraints don't bind, we can proceed as before:

$$v(a, y) = \max_{(c,a') \in \Gamma(a,y)} \left\{ u(c) + \beta \int v(a', y')Q(y, dy') \right\}$$

First order condition:

$$u'(c) = \beta \int v_a(a', y')Q(y, dy')$$

Envelope condition:

$$v_a(a, y) = Ru'(c)$$

So we have the (stochastic) Euler equation:

$$u'(c(a, y)) = \beta R \int u'(c'(a'(a, y), y'))Q(y, dy')$$

Or:

$$u'(c_t) = \beta R E_t u'(c_{t+1})$$