Lecture 11: Consumption-Savings Problem

Under Uncertainty

Economics 714, Fall 2014

1 Consumption-Savings Problem under Uncertainty

1.1 Basic Problem

\[
\max_{\{c_t, a_{t+1}\}} \sum_{t=0}^{\infty} \beta^t u(c_t) \\
\text{s.t. } c_t + a_{t+1} = Ra_t + y_t
\]

\(a_0, y_0\) given. Slight timing change from previously.

Constraints: \(c_t \geq 0, \ a_t \geq a\). Debt limit.

Income \(y\) stochastic: \(y \in Y \subseteq \mathbb{R}_+\) compact, Borel \(\sigma\)-algebra \(\mathcal{Y}\).

\(y\) follows Markov process w/transition function \(Q\) on \((Y, \mathcal{Y})\).

Assume \(Q\) has Feller property: for \(f : Y \rightarrow \mathbb{R}\) bounded, continuous then:

\[
E[f(y')|y] = \int f(y')Q(y, dy')
\]
is bounded and continuous

State space: assets \(a \in A = [a, \infty), \) Borel \(\sigma\)-algebra \(\mathcal{A}\).

Joint state space \((a, y) \in X = A \times Y\) with Borel \(\sigma\)-algebra \(\mathcal{X}\)

Feasible correspondence:

\[
\Gamma(x) = \{(c, a') : c + a' \leq Ra + y, \ c \geq 0, \ a' \geq a\}
\]
1.2 Sequence Problem

At each date $c_t : Y^t \to \mathbb{R}_+$, measurable (w.r.t $Y^t$).
$a_{t+1} : Y^t \to \mathbb{R}_+$, measurable (w.r.t $Y^t$)

$$v^*(a_0, y_0) = \sup_{\{c_t(Y^t), a_{t+1}(Y^t)\}} \left[ E_0 \sum_{t=0}^{\infty} \beta^t u(c_t) \right]$$

$$= \sup_{\{c_t(Y^t), a_{t+1}(Y^t)\}} \left[ \int \sum_{t=0}^{\infty} \beta^t u(c_t(Y^t)) Q(y_t, dy_{t+1}) Q(y_{t-1}, dy_t) \cdots Q(y_0, dy_1) \right]$$

1.3 Bellman Equation

$$v(a, y) = \max_{(c, a') \in \Gamma(a, y)} \left\{ u(c) + \beta \int v(a', y') Q(y, dy') \right\}$$

Extensions of the previous results apply, principle of optimality is direct.

Define Bellman operator as before:

$$Tf(a, y) = \max_{(c, a') \in \Gamma(a, y)} \left\{ u(c) + \beta \int f(a', y') Q(y, dy') \right\}$$

**Theorem:** Under the assumptions here $T : C(X) \to C(X)$ is a contraction, and hence has a unique fixed point $v \in C(X)$ and for all $v_0 \in C(X)$:

$$\|T^n v_0 - v\| \leq \beta^n \|v_0 - v\|$$

Moreover, the optimal policy correspondence:

$$G(x) = \{(c, a') \in \Gamma(a, y) : v(a, y) = u(c) + \beta \int v(a', y') Q(y, dy')\}$$

is compact-valued and uhc.
In addition, under our standing assumptions we have the stronger results.

**Theorem:**
(i) $v(a, y)$ is strictly increasing in $a$

(ii) $v(a, y)$ is strictly concave in $a$ and the optimal policy functions $c(a, y)$ and $a'(a, y)$ are continuous

(iii) If $(a_0, y_0) \in \text{int}(X)$ and $(c(a_0, y_0), a'(a_0, y_0)) \in \text{int}\Gamma(a_0, y_0)$ then $v$ is continuously differentiable (in $a$) at $(a_0, y_0)$ and:

$$v_a(a_0, y_0) = Ru'(c(a_0, y_0)) = Ru'(Ra + y - a'(a_0, y_0))$$

### 1.4 Euler Equations

If the constraints don’t bind, we can proceed as before:

$$v(a, y) = \max_{(c, a') \in \Gamma(a, y)} \left\{ u(c) + \beta \int v(a', y')Q(y, dy') \right\}$$

**First order condition:**

$$u'(c) = \beta \int v_a(a', y')Q(y, dy')$$

**Envelope condition:**

$$v_a(a, y) = Ru'(c)$$

So we have the (stochastic) Euler equation:

$$u'(c(a, y)) = \beta R \int u'(c'(a'(a, y), y'))Q(y, dy')$$

Or:

$$u'(c_t) = \beta RE_t u'(c_{t+1})$$