# Lecture 10: Consumption-Savings Problem

# Optimal Growth

### Economics 712, Fall 2014

## 1 Consumption-Savings Problem: Infinite Horizon

### 1.1 Basic Problem

$$\max_{\{c_t\}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to:

$$x_{t+1} = R(x_t - c_t + y_t)$$

Can formulate in SLP terms, noting $c_t = x_t + y_t - x_{t+1}/R$.

Conditions easy to verify if $u$ bounded, $u' > 0, u'' < 0$.

Write choice variable as savings $s = x - c + y$:

$$V(x) = \max_s \{u(x + y - s) + \beta V(Rs)\}$$

First-order condition, $s = g(x)$:

$$u'(x + y - g(x)) = \beta R V'(Rg(x))$$

Envelope condition:

$$V'(x) = u'(x + y - g(x))$$

Combine to get (functional) consumption Euler equation:

$$u'(x + y - g(x)) = \beta R u'(Rg(x) + y - g(g(x)))$$

### 1.2 A Solvable Example

Set $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$, $y = 0$.

Guess $V(x) = \frac{A}{1-\gamma} x^{1-\gamma}$

$$Ax^{1-\gamma} = \max_s \left\{ \frac{(x-s)^{1-\gamma}}{1-\gamma} + \beta A (Rs)^{1-\gamma} \right\}$$

Can verify that:

$$\begin{aligned} A &= \left(1 - \beta^{\frac{1}{\gamma}} R^{\frac{1-\gamma}{\gamma}}\right)^{-\gamma} \\ s &= kx = \beta^{\frac{1}{\gamma}} R^{\frac{1-\gamma}{\gamma}} \end{aligned}$$

# 2 Review: Optimal Growth

Planner's problem: Neoclassical, Ramsey-Cass-Koopmans optimal growth model

$$\max_{\{c_t, k_{t+1}\}} \sum_{t=0}^{\infty} \beta^t U(c_t)$$

subject to:

$$c_t + k_{t+1} - (1-\delta) k_t = F(k_t)$$

Bellman equation:

$$V(k) = \max_{c, k'} \{ u(c) + \beta V(k') \}$$

subject to

$$c + k' - (1-\delta) k = F(k)$$

First order and envelope conditions:

$$
\begin{aligned}
0 &= -U'(f(k) + (1-\delta)k - k') + \beta V'(k') \\
V'(k) &= U'((f(k) + (1-\delta)k - k'))(F'(k) + 1 - \delta)
\end{aligned}
$$

Combining gives the Euler equation:

$$
U'(f(k) + (1-\delta)k - k') = \beta U'((F(k') + (1-\delta)k' - k''))(F'(k') + 1 - \delta)
$$

Or reintroducing $c$ as an (endogenous) variable gives the system:

$$
\begin{aligned}
U'(c) &= \beta U'(c')(F'(k') + 1 - \delta) \\
F(k) &= c + k' - (1-\delta)k
\end{aligned}
$$

Use to study qualitative dynamics