1 Consumption-Savings Problem: Infinite Horizon

1.1 Basic Problem

\[
\max_{\{c_t\}} \sum_{t=0}^{\infty} \beta^t u(c_t)
\]

subject to:

\[x_{t+1} = R(x_t - c_t + y_t)\]

Can formulate in SLP terms, noting \( c_t = x_t + y_t - x_{t+1}/R \).

Conditions easy to verify if \( u \) bounded, \( u' > 0, u'' < 0 \).

Write choice variable as savings \( s = x - c + y \):

\[V(x) = \max_s \{u(x + y - s) + \beta V(Rs)\}\]

First-order condition, \( s = g(x) \):

\[u'(x + y - g(x)) = \beta RV'(Rg(x))\]

Envelope condition:

\[V'(x) = u'(x + y - g(x))\]

Combine to get (functional) consumption Euler equation:

\[u'(x + y - g(x)) = \beta Ru'(Rg(x) + y - g(g(x)))\]
1.2 A Solvable Example

Set \( u(c) = \frac{c^{1-\gamma}}{1-\gamma}, \) \( y = 0. \)

Guess \( V(x) = \frac{A}{1-\gamma}x^{1-\gamma} \)

\[
Ax^{1-\gamma} = \max_s \left\{ \frac{(x-s)^{1-\gamma}}{1-\gamma} + \beta A(Rs)^{1-\gamma} \right\}
\]

Can verify that:

\[
A = \left(1 - \beta^\frac{1}{\gamma}R^{\frac{1-\gamma}{\gamma}}\right)^{-\gamma}
\]

\[
s = kx = \beta^\frac{1}{\gamma}R^{\frac{1-\gamma}{\gamma}}
\]

2 Review: Optimal Growth

Planner’s problem: Neoclassical, Ramsey-Cass-Koopmans optimal growth model

\[
\max_{\{c,t,k\}} \sum_{t=0}^{\infty} \beta^t U(c_t)
\]

subject to:

\[
c_t + k_{t+1} - (1 - \delta)k_t = F(k_t)
\]

Bellman equation:

\[
V(k) = \max_{ck'} \{u(c) + \beta V(k')\}
\]

subject to

\[
c + k' - (1 - \delta)k = F(k)
\]
First order and envelope conditions:

\[ 0 = -U'(f(k) + (1 - \delta)k - k') + \beta V'(k') \]

\[ V'(k) = U'((f(k) + (1 - \delta)k - k'))(F'(k) + 1 - \delta) \]

Combining gives the Euler equation:

\[ U'(f(k) + (1 - \delta)k - k') = \beta U'(c' + (1 - \delta)k' - k'')(F'(k') + 1 - \delta) \]

Or reintroducing \( c \) as an (endogenous) variable gives the system:

\[ U'(c) = \beta U'(c') (F'(k') + 1 - \delta) \]

\[ F(k) = c + k' - (1 - \delta)k \]

Use to study qualitative dynamics