Lecture 10/11
Dynamics of the Growth Model

Noah Williams

University of Wisconsin - Madison

Economics 712
Fall 2014

Optimal Steady State

- Look for a steady state of the optimal allocation.

$$U'(c^*) = \beta U'(c^*)[F'(k^*) + 1 - \delta]$$

Or, defining that $\beta = 1/(1+\theta)$:

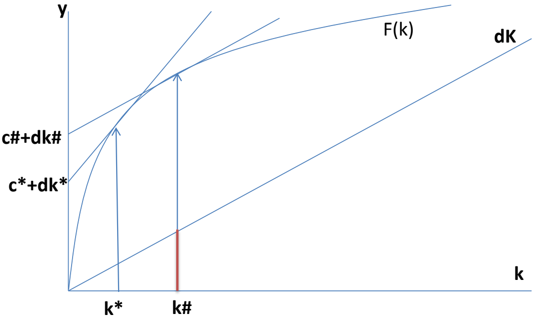
$$\begin{aligned} F'(k^*) &= \frac{1}{\beta} + \delta - 1 = \delta + \theta \\ c^* &= F(k^*) - \delta k^* \end{aligned}$$

- The "golden rule" maximizes steady state consumption:

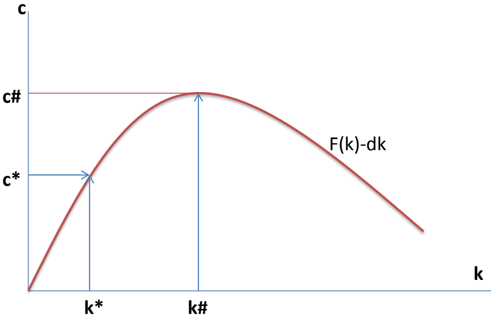
$$\max_k U(F(k) - \delta k) \Rightarrow F'(k^\#) = \delta$$

- The optimal steady state is only equal to the golden rule if $\theta = 0$. And since $F''(k) < 0$ we have:

$$F'(k^\#) = \delta < \delta + \theta = F'(k^*), \ \Rightarrow k^\# > k^*$$



Optimal steady state consumption and capital



Optimal steady state and golden rule

An Example

- Now work out a parametric example, using standard functional forms. Cobb-Douglas production:

$$y = Ak^{\alpha}$$

- For preferences, set:

$$U(c) = \frac{c^{1-\sigma} - 1}{1-\sigma}$$

For $\sigma > 0$. Interpret $\sigma = 1$ as $U(c) = \log c$.

- These imply the Euler equation:

$$c_t^{-\sigma} = \beta c_{t+1}^{-\sigma}[1 + \alpha A k_{t+1}^{\alpha-1} - \delta] = \beta c_{t+1}^{-\sigma} R_{t+1}$$

- For these preferences σ gives the curvature and so governs how the household trades off consumption over time.

Intertemporal Elasticity of Substitution

- Define intertemporal elasticity of substitution IES as:

$$IES = \frac{d\frac{c_{t+1}}{c_t}}{dR_{t+1}} \frac{R_{t+1}}{\frac{c_{t+1}}{c_t}} = \frac{d\log\left(\frac{c_{t+1}}{c_t}\right)}{d\log R_{t+1}}$$

- Then for these preferences we have:

$$\begin{aligned}
c_t^{-\sigma} &= \beta c_{t+1}^{-\sigma} R_{t+1} \\
\Rightarrow \left(\frac{c_{t+1}}{c_t}\right)^{\sigma} &= \beta R_{t+1} \\
\Rightarrow \frac{c_{t+1}}{c_t} &= \beta^{\frac{1}{\sigma}} R_{t+1}^{\frac{1}{\sigma}} \\
\Rightarrow \log\left(\frac{c_{t+1}}{c_t}\right) &= \frac{1}{\sigma}\log\beta + \frac{1}{\sigma}\log(R_{t+1}) \\
\Rightarrow IES &= \frac{1}{\sigma}
\end{aligned}$$

Steady State in the Example

- Recall the Euler equation:

$$c_t^{-\sigma} = \beta c_{t+1}^{-\sigma}[1 + \alpha A k_{t+1}^{\alpha-1} - \delta]$$

- Steady state:

$$\begin{aligned} F'(k^*) &= A\alpha(k^*)^{\alpha-1} = \delta + \theta \\ \Rightarrow k^* &= \left(\frac{\alpha A}{\delta + \theta}\right)^{\frac{1}{1-\alpha}} \end{aligned}$$

- Then we get consumption:

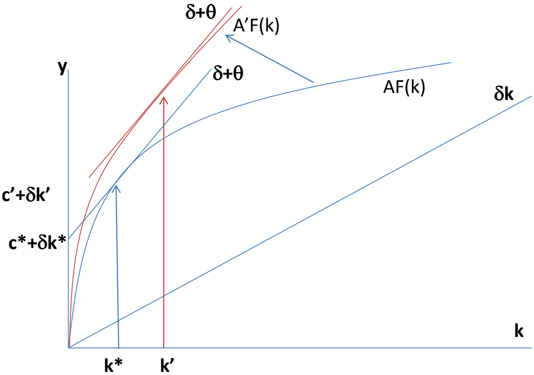
$$\begin{aligned} c^* &= A(k^*)^{\alpha} - \delta k^* \\ &= A\left(\frac{\alpha A}{\delta + \theta}\right)^{\frac{\alpha}{1-\alpha}} - \delta\left(\frac{\alpha A}{\delta + \theta}\right)^{\frac{1}{1-\alpha}} \end{aligned}$$

Comparative Statics

- Steady state capital stock determined by:

$$F'(k^*) = \delta + \theta$$

  Consumption c^* increasing in k^* (since below golden rule level).
- If $\delta \uparrow$, then $k^* \downarrow$ so $c^* \downarrow$.
- If TFP $\uparrow$ then $k^* \uparrow$, so $c^* \uparrow$.



An increase in total factor productivity

Qualitative Dynamics

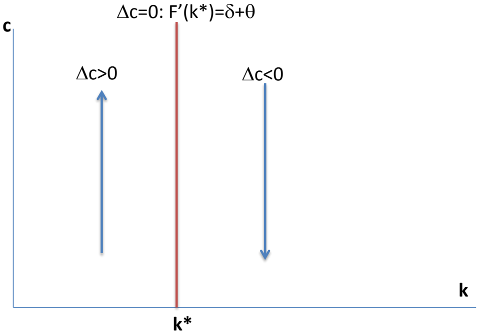
- We will analyze the joint dynamics of $\{c_t, k_t\}$. In any period t, k_t is given and c_t is chosen optimally (as is k_{t+1}).
- The key equations of the model are:

$$\begin{aligned} U'(c_t) &= \beta U'(c_{t+1})[F'(k_{t+1}) + 1 - \delta] \\ k_{t+1} &= (1-\delta)k_t + F(k_t) - c_t \end{aligned}$$

- We'll use the first to determine the dynamics of c, the second the dynamics of k.
- In steady state, $\Delta c_{t+1} = c_{t+1} - c_t = 0$, and

$$F'(k^*) = \delta + \theta$$

- If $k < k^*$, then $F'(k) > F'(k^*)$, so to satisfy Euler equation we need $U'(c_{t+1}) < U'(c_t)$ and so $c_{t+1} > c_t$. Similarly if $k > k^*$, $\Delta c < 0$.



Dynamics of consumption

Dynamics of Capital and Phase Diagram

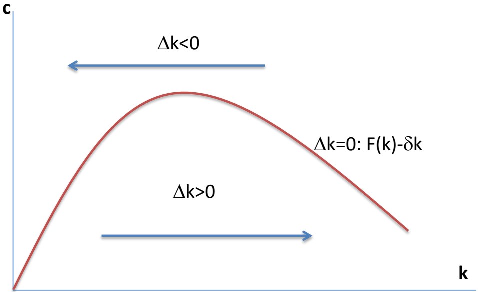
- A key equation of the model is:

$$k_{t+1} = (1-\delta)k_t + F(k_t) - c_t$$

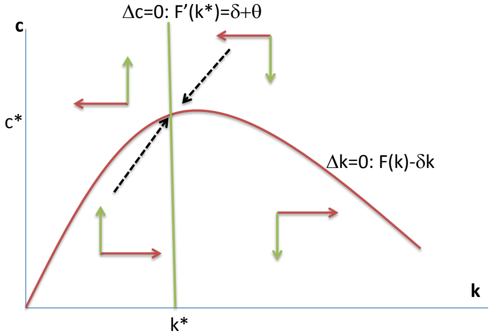
- In steady state, $\Delta k_{t+1} = k_{t+1} - k_t = 0$, and

$$c = F(k) - \delta k$$

- If $c_t < F(k_t) - \delta k_t$ then $i_t > \delta k_t$, so $\Delta k_{t+1} > 0$. Similarly if $c_t > F(k_t) - \delta k_t$ then $\Delta k_{t+1} < 0$.
- Putting together the dynamics of c with the dynamics of k gives the **phase diagram**



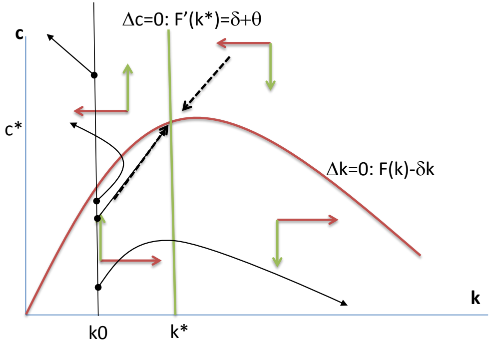
Dynamics of capital



Phase diagram: dynamics of consumption and capital

The Saddle Path

- From the phase diagram we can see the dynamics of $\{k_t, c_t\}$ from any initial (k_0, c_0).
- But given k_0 the dynamics from an arbitrary c_0 will not converge to the steady state.
- In general either c_t or k_t will go to zero. These are not part of an optimal solution.
- However given k_0 there is a unique value of c_0 such that the economy converges to the steady state. This is the **saddle path**.
- The optimal solution will be on the saddle path, as c_0 is a function of k_0 and will be chosen so that the economy is stable and converges to the steady state.



Phase diagram: saddle path and dynamics from different initial consumption levels

Comparative Dynamics

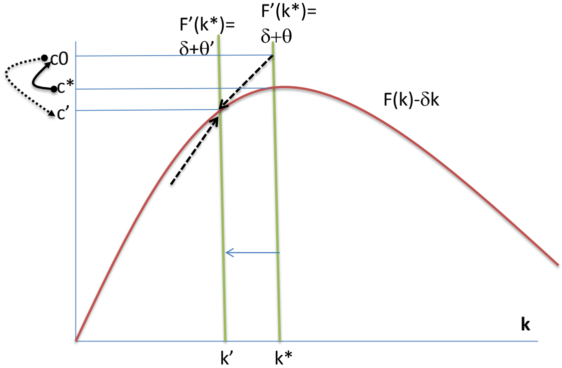
- With the phase diagram, we can determine how an exogenous change affects c_t and k_t both over time and in the long run.
- Suppose, for example, that households became less patient, so θ increases and β falls. What would happen immediately and in the long run?
- Recall the dynamics:

$$\Delta c = 0 : F'(k^*) = \delta + \theta$$

  this curve shifts to the left, so steady state k^* would fall
- And for capital,

$$\Delta k = 0 : c = F(k) - \delta k$$

  this curve is unaffected.
- In the long run, k_t and c_t will fall. When the change in θ happens c_t will jump up to the new saddle path.



Phase diagram: An increase in the discount rate (θ).

When households become more impatient, they increase consumption, and save less. In the short run this leads to more consumption. But in the long run, the lower investment will lead to a reduction in capital and hence consumption.

Improvement in Technology

- If total factor productivity increases, we already have seen that in steady state $k^* \uparrow$, so $c^* \uparrow$. But what happens along the transition?
- Recall the dynamics:

$$\Delta c = 0 : AF'(k^*) = \delta + \theta$$

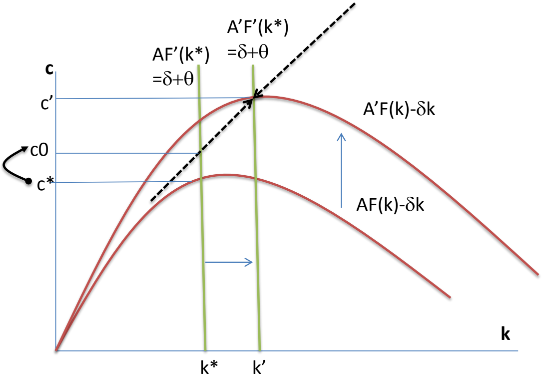
  this curve shifts to the right
- And for capital,

$$\Delta k = 0 : c = AF(k) - \delta k$$

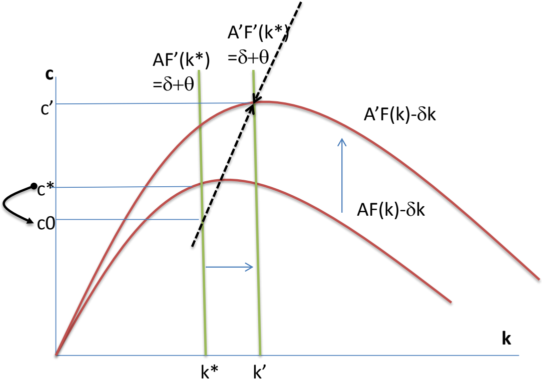
  this curve shifts up
- In the long run (k^*, c^*) increase. But the effect in the short run depends on the slope of the saddle path, which in turn depends on how willing the household is to substitute over time.

Slope of the Saddle Path

- If the household is relatively **impatient** (low β, high θ), and is **unwilling** to substitute over time (low IES), then it will want to smooth consumption over time and value early periods highly. Then c_t will increase and the economy will slowly move to the steady state. c_0 will increase.
- If the household is relatively **patient** (high β, low θ), and is **willing** to substitute over time (high IES), then it will forgo current consumption, invest more, and get to the steady state more quickly. c_0 may fall.
- In both cases, k_t increases each period until it reaches the steady state. In the long run c_t increases, but in the short-run it may increase or decrease.



Phase diagram: An increase in total factor productivity (A), with a flatter saddle path (low IES, high discount rate)



Phase diagram: An increase in total factor productivity (A), with a steep saddle path (high IES, low discount rate)

Solving the Model in a Special Case

- There is one known case where we can work out an explicit solution.
- Set $\delta = 1$ (full depreciation) use logarithmic utility, Cobb-Douglas:

$$U(c) = \log c, \quad F(k) = Ak^{\alpha}$$

- Specialize the key equilibrium equations:

$$\begin{aligned} \frac{1}{c_t} &= \frac{\beta \alpha A k_{t+1}^{\alpha-1}}{c_{t+1}} \\ c_t &= Ak_t^{\alpha} - k_{t+1} \end{aligned}$$

- Guess that the solution is a constant savings rate s:

$$c_t = (1-s)y_t$$

- Substitute into conditions:

$$\begin{aligned}
\frac{1}{(1-s)Ak_t^{\alpha}} &= \frac{\beta\alpha Ak_{t+1}^{\alpha-1}}{(1-s)Ak_{t+1}^{\alpha}} \\
&= \frac{\beta\alpha}{(1-s)k_{t+1}} \\
&= \frac{\beta\alpha}{(1-s)sAk_t^{\alpha}}
\end{aligned}$$

- So $s = \beta\alpha$, and $c_t = (1-\beta\alpha)Ak_t^{\alpha}$.

Implications

- In this special case we have the explicit relationship between (c, k), the optimal decision rule or the saddle path.
- We then have the dynamics of k_t:

$$k_{t+1} = sAk_t^{\alpha}$$

The steady state is a special case of what we had earlier $(\delta = 1)$:

$$k^* = \left(\frac{\alpha A}{1+\theta}\right)^{\frac{1}{1-\alpha}}$$

- So we can now trace out the dynamics explicitly. For example, if A increases, c_t increases on impact and grows over time to the new steady state.