Household Problem

\[
\max_{c, c'} u(c) + \beta u(c') \quad \text{s.t. } c + \frac{c'}{1 + r} = y^{PV}
\]

- Form Lagrangian with multiplier \( \lambda > 0 \).

\[
L = u(c) + \beta u(c') + \lambda \left( y^{PV} - c - \frac{c'}{1 + r} \right)
\]

FOC: \( u'(c) = \lambda \)

\( \beta u'(c') = \frac{\lambda}{1 + r} \)

- Combine them to get Euler Equation:

\[
u'(c) = \beta (1 + r) u'(c')
\]
What happens if $y$, $y'$ or $A$ increases? All matters is $y^{PV}$.

- Both $c$ and $c'$ increase (normal goods).

- If $y$ or $A$ increase, $s$ increases to finance higher $c'$.
  Examples: increases in stock market or house prices – “wealth effect”

- If $y'$ increases, $s$ falls to finance higher current $c$.
  Examples: Announced layoffs, changing professions (or college majors).

- Sometimes discuss marginal propensity to consume (MPC).
  For the log example, MPC out of current income or wealth:

\[
    c = \frac{y^{PV}}{1 + \beta} \\
    \frac{\partial c}{\partial A} = \frac{\partial c}{\partial y} = \frac{1}{1 + \beta} > 0
\]
Figure 8.5  The Effects of an Increase in Current Income for a Lender
Figure 8.9  Stock Prices and Consumption of Nondurables and Services, 1985–2006

Source: Standard and Poor's; Bureau of Economic Analysis, Department of Commerce.
Figure 8.10 Scatter Plot of Percentage Deviations from Trend in Consumption of Nondurables and Services Versus Percentage Deviations from Trend in a Stock Price Index

Source: Standard and Poor’s; Bureau of Economic Analysis, Department of Commerce.
**Income effect:** if a saver $s > 0$, then higher interest rate increases income for given amount of saving. Increases consumption in first and second period. If borrower $s < 0$, then income effect negative.

**Substitution effect:** gross interest rate $1 + r$ is relative price of consumption in period 1 to consumption in period 2. Current $c$ becomes more expensive relative to $c'$. This increases $c'$ and reduces $c$.

**Hence:** for a saver an increase in $r$ increases $c'$ and may increase or decrease $c$. For a borrower an increase in $r$ reduces $c$ and may increase or decrease $c'$. 
Figure 8.12 An Increase in the Real Interest Rate for a Lender
Figure 8.13  An Increase in the Real Interest Rate for a Borrower
Infinite Horizon Model

- Now extend the consumption-savings model from 2 periods to an infinite horizon. Many of the same implications.
- Slightly different timing/notation following Wickens.
- Flow budget constraint: $c_t$ consumption at date $t$, $a_t$ assets on hand at start of $t$. $a_{t+1}$ assets chosen at $t$, carried over to $t+1$, $r_t$ interest rate between $t-1$ and $t$, $x_t$ income:

$$c_t + a_{t+1} = x_t + (1 + r_t) a_t$$

- Derive intertemporal budget constraint, with $r_0 = 0$:

$$c_0 = x_0 - a_1 + a_0$$
$$= x_0 - \frac{c_1 - x_1}{1 + r_1} - \frac{a_2}{1 + r_1} + a_0$$
$$= x_0 - \frac{c_1 - x_1}{1 + r_1} - \frac{c_2 - x_2}{(1 + r_1)(1 + r_2)} - \frac{a_3}{(1 + r_1)(1 + r_2)} + a_0$$

$$c_0 + \frac{c_1}{1 + r_1} + \frac{c_2}{(1 + r_1)(1 + r_2)} =$$
$$x_0 + \frac{x_1}{1 + r_1} + \frac{x_2}{(1 + r_1)(1 + r_2)} - \frac{a_3}{(1 + r_1)(1 + r_2)} + a_0$$
Intertemporal Budget Constraint

- Continue same process for any horizon $T$:

$$
\sum_{t=0}^{T} \frac{c_t}{\prod_{s=0}^{t}(1 + r_s)} = \sum_{t=0}^{T} \frac{x_t}{\prod_{s=0}^{t}(1 + r_s)} + a_0 - \frac{a_{T+1}}{\prod_{s=0}^{T}(1 + r_s)}
$$

- For any finite horizon $T$ we would have $a_{T+1} = 0$. No reason to save, and more importantly no one would lend.

- For infinite horizon, need to rule out the possibility of borrowing forever and never repaying principal.

- A **Ponzi game** occurs when agents borrow, repaying existing debt obligations by borrowing more. We impose the No Ponzi Game (NPG) restriction:

$$
\lim_{T \to \infty} \frac{a_{T+1}}{\prod_{s=0}^{T}(1 + r_s)} \geq 0
$$

- This rules out borrowing indefinitely. Household won’t want to have strictly positive assets in limit, so NPG will hold with equality.
Household Problem: Infinite Horizon

- Under the NPG restriction we can take limits as $T \to \infty$:
  \[
  \sum_{t=0}^{\infty} \frac{c_t}{\prod_{s=0}^{t}(1 + r_s)} = \sum_{t=0}^{\infty} \frac{x_t}{\prod_{s=0}^{t}(1 + r_s)} + a_0 \equiv x^{PV}
  \]

- The household problem is now to choose $\{c_t\}_{t=0}^{\infty}$ to maximize utility subject to the present value budget constraint. Single optimization problem, choosing plan for consumption for entire future.

- Lagrangian:
  \[
  L = \sum_{t=0}^{\infty} \beta^t u(c_t) + \lambda \left( x^{PV} - \sum_{t=0}^{\infty} \frac{c_t}{\prod_{s=0}^{t}(1 + r_s)} \right)
  \]
Household Problem: Optimality Conditions

- First order condition for consumption at any dates $t, t + 1$:

$$\beta^t u'(c_t) = \frac{\lambda}{\prod_{s=0}^{t}(1 + r_s)}$$

$$\beta^{t+1} u'(c_{t+1}) = \frac{\lambda}{\prod_{s=0}^{t+1}(1 + r_s)}$$

- Divide these two equations:

$$\frac{u'(c_t)}{\beta u'(c_{t+1})} = \frac{\prod_{s=0}^{t+1}(1 + r_s)}{\prod_{s=0}^{t}(1 + r_s)} = 1 + r_{t+1}$$

- So once again we get the consumption Euler equation:

$$u'(c_t) = \beta u'(c_{t+1})(1 + r_{t+1})$$

- This governs behavior of consumption for any dates $t, t + 1$. 

---

Williams  Economics 312/702
One application: Franco Modigliani’s life-cycle hypothesis of consumption

Individuals want smooth consumption profile over their life. Labor income varies substantially over lifetime, starting out low, increasing until around the 50th year of a person’s life and then declining until retirement around 65, with no labor income after retirement.

Life-cycle hypothesis: by saving and borrowing individuals turn a very non-smooth labor income profile into a very smooth consumption profile.
Suppose that $r_t = r \ \forall t$, and $\beta(1 + r) = 1$. Then Euler equation implies $c_t = c_{t+1} = \bar{c}$.

Use present value budget constraint to work out consumption level:

$$
\sum_{t=0}^{\infty} \frac{c_t}{(1 + r)^t} = x^{PV} 
$$

$$
\Rightarrow \bar{c} \sum_{t=0}^{\infty} \frac{1}{(1 + r)^t} = \frac{\bar{c}(1 + r)}{r} = x^{PV} 
$$

So $c_t = \frac{r}{1+r} x^{PV}$ for all $t$.

If $x_t = \frac{r}{1+r} x^{PV}$ for all $t$ then $a_t = 0$ for all $t$. 
In general, consumption is constant but income $x_t$ varies. How is this implemented?

$$c_0 = x_0 - a_1 + a_0 \Rightarrow a_1 = x_0 - c_0 + a_0$$

$$a_1 = x_0 + a_0 - \frac{r}{1+r} x^{PV}$$

If current income $x_0 + a_0$ is low relative to $\frac{r}{1+r} x^{PV}$, borrow $a_1 < 0$.
If $x_0 + a_0$ is high relative to $\frac{r}{1+r} x^{PV}$, save $a_1 > 0$.

These same general implications extend to varying $r_t$, $\beta(1 + r_t) \neq 1$.

Main predictions: current consumption depends on total lifetime income and initial wealth. Saving should follow a very pronounced life-cycle pattern with borrowing in the early periods of an economic life, significant saving in the high earning years from 35-50 and dissaving in retirement years.
Figure 4.A.5
Life-cycle consumption, income, and saving
This pattern of life-cycle savings is generally consistent with the data.

One empirical puzzle: Older households do not dissave to the extent predicted by the theory. Several explanations:

1. Individuals are altruistic and want to leave bequests to their children.
2. Uncertainty with respect to length of life and health status.

Important in aggregate as population ages (Japan).
Japanese saving rate fell from 23% of personal income in 1975 to 14% in 1990 down to 5% in 2000.
Over same horizon, US saving rate roughly flat around 6%.
Ratio of Japanese over age of 65 to those of working age rose from 15% in 1980 to 28% in 2000. Forecast to increase further to 38% by 2010 and 50% by 2020.
Estimates by HSBC that demographic shift can account for half of the decline in the savings rate.
Effects of inflation, slower growth rates, changes in government debt are other factors contributing to savings decline.
FIGURE 1. NET NATIONAL SAVING RATES
Future income is uncertain.

Income of an individual household, $x_t$ consists of a permanent part, $x^p$ and a transitory part $v_t$

$$x_t = x^p + v_t$$

Permanent part $x^p$: expected average future income (usual salary)

Transitory part $v_t$: random fluctuations around this average income (bonus)

In two period model from last time, permanent means $y$ and $y'$ change. Transitory: only $y$ changes.
Friedman (1956): Individuals react differently to an increase in permanent and an increase in transitory income. Increase in the permanent component of income brings about an (almost) equal response in consumption. Large increase in $x^{PV}$.

Individuals smooth out transitory income shocks over time. Little effect on $x^{PV}$. Greater fraction of increase is saved.

It follows that individual consumption is almost entirely determined by permanent income. So consumption should be smoother than income.

Data suggests it is so, but not as smooth as theory suggests. Effects of credit market imperfections and borrowing constraints.
Figure 8.8  Temporary Versus Permanent Increases in Income
Now $x_{t+1}, r_{t+1}$ are random, unknown at $t$.

Agents form expectations of future income, maximize expected utility.

Can derive an Euler equation of the same form, but now must have expectations over $c_{t+1}$ and $r_{t+1}$:

$$u'(c_t) = \beta E_t [u'(c_{t+1})(1 + r_{t+1})]$$

Here $E_t(\cdot)$ represents the agent’s expectations, conditional on all information available at date $t$. 
Suppose again that $r_t = r$ and $\beta(1 + r) = 1$, so the Euler equation is:

$$u'(c_t) = E_t u'(c_{t+1})$$

Also suppose that agents have quadratic preferences, where $a > 0$ is a constant:

$$u(c_t) = c_t - \frac{a}{2} c_t^2,$$

So $u'(c_t) = 1 - ac_t$ and the Euler equation becomes:

$$c_t = E_t c_{t+1}$$

Also by the law of iterated expectations:

$$c_t = E_t c_{t+1} = E_t (E_{t+1} c_{t+2}) = E_t c_{t+2}$$
With these preferences consumption is a random walk:

\[ c_{t+1} = c_t + \varepsilon_{t+1}, \quad E_t\varepsilon_{t+1} = 0 \]

The best predictor of consumption one period ahead is current consumption. No other variables which are known at date \( t \) help predict consumption at \( t + 1 \).

To express this another way, note that the present value budget constraint holds for any date \( t \):

\[
\sum_{s=0}^{\infty} \frac{E_t c_{t+s}}{(1 + r)^s} = \sum_{s=0}^{\infty} \frac{E_t x_{t+s}}{(1 + r)^s} + a_t(1 + r)
\]
Then note that $E_t c_{t+s} = c_t$ for all $s$. So then we have:

$$
c_t \sum_{s=0}^{\infty} \frac{1}{(1 + r)^s} = \sum_{s=0}^{\infty} \frac{E_t x_{t+s}}{(1 + r)^s} + a_t(1 + r)
$$

$$
c_t = \frac{r}{1 + r} \sum_{s=0}^{\infty} \frac{E_t x_{t+s}}{(1 + r)^s} + r a_t
$$

Consumption depends on expectations of all future income.

Changes in consumption over time are driven by changes in expectations of future income. Information revealed about future income is the driver of consumption.

$$
c_{t-1} = E_{t-1} c_t = \frac{r}{1 + r} \sum_{s=0}^{\infty} \frac{E_{t-1} x_{t+s}}{(1 + r)^s} + r a_t
$$
A pure transitory income shock reveals at date $t$ that $x_t > E_{t-1}x_t$ is higher than anticipated, but $E_t x_{t+s}$ is unaffected for $s \geq 1$. Example: $x_t = x_{t-1} + v_t$, $x_{t+s} = x_{t-1}$.

$$c_t = c_{t-1} + \frac{r}{1 + r} v_t$$

A permanent income shock reveals at date $t$ that $x_t > E_{t-1}x_t$ is higher than anticipated, and $E_t x_{t+s}$ is also higher for $s \geq 1$. Example: $x_{t+s} = x_{t-1} + x^p$

$$c_t = c_{t-1} + x^p$$