• Look for a steady state of the transformed optimal allocation.

\[(1 + \eta)(\tilde{c}^*)^{-\sigma} = \tilde{\beta}(\tilde{c}^*)^{-\sigma}[\alpha(\tilde{k}^*)^{\alpha-1} + 1 - \delta]\]

\[(1 + \eta) = \tilde{\beta}[\alpha(\tilde{k}^*)^{\alpha-1} + 1 - \delta]\]

• Or, recalling that \(\beta = 1/(1 + \theta)\):

\[f'(\tilde{k}^*) = \frac{1 + \eta}{\beta(1 + \eta)^{1-\sigma}} + \delta - 1\]

\[= \frac{(1 + \theta)}{(1 + \eta)^{-\sigma}} + \delta - 1\]

\[\approx \delta + \theta + \sigma \eta\]

\[\approx \delta + \theta + \sigma(n + g)\]
Therefore we have capital per unit of effective labor in the balanced growth path:

\[
\tilde{k}^* = \left( \frac{\alpha}{\delta - 1 + (1 + \theta)(1 + \eta)^\sigma} \right)^{\frac{1}{1-\alpha}} \\
\approx \left( \frac{\alpha}{\delta + \theta + \sigma(n + g)} \right)^{\frac{1}{1-\alpha}}
\]

This generalizes the solution we had for the optimal allocation without growth.

As in the Solow model, along a balanced growth path all level variables are growing at rate \( \eta \approx n + g \).

Unlike the Solow model, the steady state depends on the household preferences, as the savings rates are determined optimally.
We can analyze the qualitative dynamics just as we did without productivity growth.

The key equations of the model are now:

\[
U'(\tilde{c}_t) = \beta (1 + \eta)^{1-\sigma} U'(\tilde{c}_{t+1}) [f'(\tilde{k}_{t+1}) + 1 - \delta]
\]

\[
(1 + \eta)\tilde{k}_{t+1} = (1 - \delta)\tilde{k}_t + f(\tilde{k}_t) - \tilde{c}_t
\]

The dynamics work in much the same way, only now they depend on \(\eta\). So we can analyze the effects of a change in \(n\) or \(g\) which lead to a change in \(\eta\).

In steady state, \(\Delta \tilde{c}_{t+1} = 0\), and

\[
f'(\tilde{k}^*) \approx \delta + \theta + \sigma \eta
\]

Also in steady state \(\Delta \tilde{k}_{t+1} = 0\), so:

\[
\tilde{c} = f(\tilde{k}) - (\delta + \eta)\tilde{k}
\]
Phase diagram of the optimal growth model

\[ \Delta c = 0 : f'(k^*) = \delta + \theta + \sigma \eta \]

\[ \Delta k = 0 : f(k) - (\delta + \eta)k \]

Phase diagram of the optimal growth model
Effect of an Increase in \( n \) or \( g \)

Phase diagram: An increase in the growth rate \( \eta \) to \( \eta' \). As before, initial effect depends on the slope of the saddle path.
Now briefly discuss some models which try to explain sources of growth *endogenous growth models*.


More recently Acemoglu et al: role of institutions in growth.
Aside from innovations (which we’ll turn to next), infrastructure, institutions, and geography are also important.


Small initial differences in income.

Differences in settlers mortality influenced whether colony was run for “extraction” or whether colonists developed institutions. Those colonies where institutions took hold developed faster.

Large differences in outcomes – still today!
Figure 1. Reduced-Form Relationship Between Income and Settler Mortality
Relatively new branch of economic theory: endogenous growth theory seeks to explain how technical change happens.

Simple endogenous growth model (AK model): aggregate production function $Y = AK$. (Ignore labor and population growth, could think of this as per capita production.)

Not subject to diminishing returns: MPK is constant

$$F_K = \frac{Y}{K} = A.$$ 

Idea: Aggregate capital K captures not just increases in physical capital but changes in the makeup of that capital.
Human capital: knowledge, skills, and training of individuals. As economies become richer they invest in human capital in the same proportion, offsetting the diminishing marginal product of physical capital alone.

Explicitly: production depends on human capital $H$, physical capital $K$:

$$Y = zH^\theta K^{1-\theta}$$

Say $H = hK$, so that human capital is constant fraction of physical, then letting $A = zh^\theta$:

$$Y = z(hK)^\theta K^{1-\theta} = \left[zh^\theta\right]K = AK$$
Research and development programs are part of capital investment. They increase the stock of knowledge, which offsets diminishing marginal products of capital accumulation.

Learning by doing: as economies produce more they learn better how to produce.
Implications of the Endogenous Growth Model

- Again savings constant fraction \( s \) of output. So:

\[
\dot{K} = sAK - \delta K
\]

- Since \( Y = AK \),

\[
\frac{\dot{Y}}{Y} = \frac{\dot{K}}{K} = sA - \delta
\]

- Growth of output depends on the saving rate, even in the long run. No steady state.

- Higher savings \( \Rightarrow \) more human capital, R&D, learning by doing. So higher savings leads to productivity improvements and higher growth.

- Important implication, some evidence that measured TFP does depend on savings, human capital.
Cobb-Douglas aggregate production function:

\[ Y = K^\alpha H^\beta (AN)^{1-\alpha-\beta} \]

Again we have constant returns to scale now in \((K, H, N)\).

Human capital and labor enter with different coefficients.

Society accumulates human capital according to:

\[ \dot{H} = s_h Y - \delta H \]

Capital accumulation equation:

\[ \dot{K} = s_k Y - \delta K \]

Technological progress: \( \frac{\dot{A}}{A} = g > 0 \).

Labor force grows at constant rate: \( \frac{\dot{N}}{N} = n > 0 \).
Dividing the production function by $AN$:

$$\tilde{y} = \tilde{k}^\alpha \tilde{h}^\beta$$

Decreasing returns to scale in per efficiency units.

The evolution of inputs is determined by:

$$\dot{\tilde{k}} = s_k \tilde{k}^\alpha \tilde{h}^\beta - (n + g + \delta)\tilde{k}$$

$$\dot{\tilde{h}} = s_h \tilde{k}^\alpha \tilde{h}^\beta - (n + g + \delta)\tilde{h}$$

System of two differential equations determining $\tilde{k}$, $\tilde{h}$. 
Balanced Growth Path

- To find the BGP equate both equations to zero:

\[ s_k \tilde{k}^* \alpha \tilde{h}^* \beta - (n + g + \delta) \tilde{k}^* = 0 \]
\[ s_h \tilde{k}^* \alpha \tilde{h}^* \beta - (n + g + \delta) \tilde{h}^* = 0 \]

- From first equation:

\[ \tilde{h}^* = \left( \frac{(n + g + \delta) \tilde{k}^* 1 - \alpha}{s_k} \right)^{\frac{1}{\beta}} \]

- Plugging it in the second equation

\[ s_h \tilde{k}^* \alpha \frac{(n + g + \delta) \tilde{k}^* 1 - \alpha}{s_k} - (n + g + \delta) \left( \frac{n + g + \delta \tilde{k}^* 1 - \alpha}{s_k} \right)^{\frac{1}{\beta}} = 0 \Rightarrow \]
\[ \frac{s_h \tilde{k}^*}{s_k} = \left( \frac{n + g + \delta \tilde{k}^* 1 - \alpha}{s_k} \right)^{\frac{1}{\beta}} \]
Finding the Balanced Growth Path

\[
\frac{sh}{sk} \tilde{k}^* = \left( \frac{n + g + \delta}{sk} \tilde{k}^{*1-\alpha} \right)^{\frac{1}{\beta}} \Rightarrow
\]

\[
\tilde{k}^{*1-\frac{1-\alpha}{\beta}} = \tilde{k}^{*1-\frac{1-\alpha-\beta}{\beta}} = \frac{sk}{sh} \left( \frac{n + g + \delta}{sk} \right)^{\frac{1}{\beta}} \Rightarrow
\]

\[
\tilde{k}^* = \left( \frac{sk^{1-\beta}sh^\beta}{n+g+\delta} \right)^{\frac{1}{1-\alpha-\beta}}
\]

\[
\tilde{h}^* = \left( \frac{sk^\alpha sh^{1-\alpha}}{n+g+\delta} \right)^{\frac{1}{1-\alpha-\beta}}
\]
The Balanced Growth Path

- Using the production function:

\[ \tilde{y} = \frac{Y}{AN} = \tilde{k}^\alpha \tilde{h}^\beta = \left( \frac{s_k^{1-\beta} s_h^\beta}{n + g + \delta} \right)^{\frac{\alpha}{1-\alpha-\beta}} \left( \frac{s_k^\alpha s_h^{1-\alpha}}{n + g + \delta} \right)^{\frac{\beta}{1-\alpha-\beta}} \Rightarrow \]

\[ y = \frac{Y}{N} = \left( \frac{s_k^{1-\beta} s_h^\beta}{n + g + \delta} \right)^{\frac{\alpha}{1-\alpha-\beta}} \left( \frac{s_k^\alpha s_h^{1-\alpha}}{n + g + \delta} \right)^{\frac{\beta}{1-\alpha-\beta}} A \]

- Given some initial value of technology \( A_0 \) we have:

\[ y = \left( \frac{s_k^{1-\beta} s_h^\beta}{n + g + \delta} \right)^{\frac{\alpha}{1-\alpha-\beta}} \left( \frac{s_k^\alpha s_h^{1-\alpha}}{n + g + \delta} \right)^{\frac{\beta}{1-\alpha-\beta}} A_0 e^{gt} \]
Evaluating the Model

Taking logs:

\[
\log y = \log A_0 + gt - \frac{\alpha + \beta}{1 - \alpha - \beta} \log (n + g + \delta) + \\
+ \frac{\alpha}{1 - \alpha - \beta} \log s_k + \frac{\beta}{1 - \alpha - \beta} \log s_h
\]

- What if we have a lot of countries \( i = 1, ..., n \)?
- We can assume that \( \log A_0 = a + \varepsilon_i \)
- Also assume that \( g \) and \( \delta \) are constant across countries.
<table>
<thead>
<tr>
<th>Sample:</th>
<th>Non-oil</th>
<th>Intermediate</th>
<th>OECD</th>
</tr>
</thead>
<tbody>
<tr>
<td>Observations:</td>
<td>98</td>
<td>75</td>
<td>22</td>
</tr>
<tr>
<td>CONSTANT</td>
<td>6.89</td>
<td>7.81</td>
<td>8.63</td>
</tr>
<tr>
<td>(1.17)</td>
<td>(1.19)</td>
<td>(2.19)</td>
<td></td>
</tr>
<tr>
<td>ln(I/GDP)</td>
<td>0.69</td>
<td>0.70</td>
<td>0.28</td>
</tr>
<tr>
<td>(0.13)</td>
<td>(0.15)</td>
<td>(0.39)</td>
<td></td>
</tr>
<tr>
<td>ln(n + g + δ)</td>
<td>-1.73</td>
<td>-1.50</td>
<td>-1.07</td>
</tr>
<tr>
<td>(0.41)</td>
<td>(0.40)</td>
<td>(0.75)</td>
<td></td>
</tr>
<tr>
<td>ln(SCHOOL)</td>
<td>0.66</td>
<td>0.73</td>
<td>0.76</td>
</tr>
<tr>
<td>(0.07)</td>
<td>(0.10)</td>
<td>(0.29)</td>
<td></td>
</tr>
<tr>
<td>$R^2$</td>
<td>0.78</td>
<td>0.77</td>
<td>0.24</td>
</tr>
<tr>
<td>s.e.e.</td>
<td>0.51</td>
<td>0.45</td>
<td>0.33</td>
</tr>
</tbody>
</table>

Restricted regression:

| CONSTANT | 7.86 | 7.97 | 8.71 |
| (0.14) | (0.15) | (0.47) |
| ln(I/GDP) − ln(n + g + δ) | 0.73 | 0.71 | 0.29 |
| (0.12) | (0.14) | (0.33) |
| ln(SCHOOL) − ln(n + g + δ) | 0.67 | 0.74 | 0.76 |
| (0.07) | (0.09) | (0.28) |
| $R^2$ | 0.78 | 0.77 | 0.28 |
| s.e.e. | 0.51 | 0.45 | 0.32 |

Test of restriction:

| p-value | 0.41 | 0.89 | 0.97 |
| Implied α | 0.31 | 0.29 | 0.14 |
| (0.04) | (0.05) | (0.15) |
| Implied β | 0.28 | 0.30 | 0.37 |
| (0.03) | (0.04) | (0.12) |

*Note. Standard errors are in parentheses. The investment and population growth rates are averages for the period 1960–1985. (g + δ) is assumed to be 0.05. SCHOOL is the average percentage of the working-age population in secondary school for the period 1960–1985.*
Figure 1: Relation of TFP growth to saving rate

Figure 2: Relation of TFP growth to schooling rate
Figure 3: Relation of TFP growth to labor force growth rate

Figure 4: Relation of TFP growth to saving rate
Now start to analyze decentralized model, building toward dynamic general equilibrium.

Start with household consumption-savings decisions. Previously in class analyzed labor-leisure decisions. Later put them together.

Start today with two period model, extend later to infinite horizon.
A Two-Period Model of Consumption and Savings

- Household preferences:

\[ U(c, c') = u(c) + \beta u(c') \]

- (Labor) income \( y > 0 \) in the first period of life and \( y' \geq 0 \) in the second period of life.

- Initial wealth \( A \geq 0 \), say received from parents.

- Household can save part of income or initial wealth in the first period, or it can borrow against future income \( y' \).

  Interest rate on both savings and on loans is equal to \( r \). Let \( s \) denote saving.

- Budget constraint in first period:

\[ c + s = y + A \]

- Budget constraint in second period:

\[ c' = y' + (1 + r)s \]
Summing both budget constraints

\[ c + \frac{c'}{1+r} = y + \frac{y'}{1+r} + A \equiv y^{PV} \]

We have normalized the price of the consumption good in the first period to 1. Price of the consumption good in period 2 is \( \frac{1}{1+r} \), which is also the relative price of consumption in period 2, relative to consumption in period 1. Gross interest rate \( 1 + r \) is the relative price of consumption goods today to consumption goods tomorrow.

Called the present value budget constraint (PVBC).
Figure 8.1 Consumer’s Lifetime Budget Constraint

\[ c' = \text{Future Consumption} \]

\[ y' - t' \]

\[ y - t \]

\[ we(1 + r) \]

Consumer is a lender

Consumer is a borrower

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• Idea of PV extends more generally to any stream of payments or costs over time. Example: widely used in consulting to value a firm’s assets and liabilities.

• General principle: income (or cost) in future is worth less than income today.

• General formula: for future income values \( \{y_1, y_2, y_3, y_4, \ldots \} \)

\[
PV = \sum_{t=1}^{T} \frac{y_t}{(1 + r)^t}.
\]

• Distinction with discounting utility: \( \beta \) reflects subjective preference, here \( 1/(1 + r) \) objective time value of money. (In equilibrium the two are linked.)
Ex 1: Valuing a treasury bill/zero coupon bond. If I buy a treasury bill today, I get $100 in six months. 

$$PV = \frac{100}{1 + r}$$, where \( r \) is the six-month interest rate. Note interest rates and bond prices are inversely related.

Ex 2: Suppose invest $5000 in a company today, it takes 3 years to become profitable, and thereafter gives $2000 in profit for 3 years. If the interest rate is 4% is this a good investment?

$$PV = -5000 + 0 + 0 + \frac{2000}{(1.04)^3} + \frac{2000}{(1.04)^4} + \frac{2000}{(1.04)^5} = $131.46.$$ 

What if \( r = 6\% \)? Can show $$PV = -242.06.$$ 

Shows the importance of the interest rate for PV.
Another example: Lottery winners always take the immediate payment over the annuity, even though the total value is less. In recent PowerBall jackpot of $295 million, the 4 winners had option of $2.95 million a year for the next 25 years ($2.95 \times 25 = $295 million, or $73.75 million each), or an immediate $41 million.

All chose immediate payoff. Why?

The present value is higher if interest rate is greater than 5.7% (Try it.)
Back to Household Problem

\[ \max_{c,c'} u(c) + \beta u(c') \quad \text{s.t.} \quad c + \frac{c'}{1+r} = y^{PV} \]

- Form Lagrangian with multiplier \( \lambda > 0 \).

\[ L = u(c) + \beta u(c') + \lambda \left( y^{PV} - c - \frac{c'}{1+r} \right) \]

FOC: \[ u'(c) = \lambda \]

\[ \beta u'(c') = \frac{\lambda}{1+r} \]

- Combine them to get Euler Equation:

\[ u'(c) = \beta (1 + r) u'(c') \]
Figure 8.3  A Consumer Who Is a Lender

In the diagram, the consumption function $c = c^*$ is shown with a shaded area indicating the consumer's indifference curve. The points $A$, $B$, $D$, and $E$ are marked on the curve, with $A$ being the optimal consumption point. The axes represent current consumption $c$ and future consumption $we(1 + r)$. The diagram illustrates the consumer's budget constraint and the trade-off between current and future consumption.
A Parametric Example

- If \( u(c) = \log c \), Euler Equation:

\[
\frac{1}{c} = \beta (1 + r) \frac{1}{c'} \Rightarrow c' = \beta (1 + r) c
\]

- Note that

\[
c = y^{PV} - \frac{c'}{1 + r} = y^{PV} - \beta c
\]

So that:

\[
c = \frac{1}{1 + \beta} y^{PV}
\]

\[
c' = \frac{\beta (1 + r)}{1 + \beta} y^{PV}
\]

\[
s = y + A - c = \frac{\beta}{1 + \beta} (y + A) - \frac{1}{1 + \beta} \left( \frac{y'}{1 + r} \right)
\]
Comparative Statics: Income Changes

What happens if $y$, $y'$ or $A$ increases? All matters is $y^{PV}$.

Both $c$ and $c'$ increase (normal goods).

If $y$ or $A$ increase, $s$ increases to finance higher $c'$. Examples: increases in stock market or house prices—“wealth effect”

If $y'$ increases, $s$ falls to finance higher current $c$. Examples: Announced layoffs, changing professions (or college majors).

Sometimes discuss marginal propensity to consume (MPC). For example, MPC out of current income or wealth:

$$\frac{\partial c}{\partial A} = \frac{\partial c}{\partial y} = \frac{1}{1 + \beta} > 0$$
Figure 8.5 The Effects of an Increase in Current Income for a Lender

\[ c = \text{Current Consumption} \]

\[ c_1 \leq c_2 \leq c_3 \]

\[ we_1(1 + r) \]

\[ we_2(1 + r) \]

Points:
- \( A \)
- \( B \)
- \( C \)
- \( D \)
- \( E_1 \)
- \( E_2 \)
Figure 8.9  Stock Prices and Consumption of Nondurables and Services, 1985–2006

Source: Standard and Poor’s; Bureau of Economic Analysis, Department of Commerce.
Figure 8.10  Scatter Plot of Percentage Deviations from Trend in Consumption of Nondurables and Services Versus Percentage Deviations from Trend in a Stock Price Index

Source: Standard and Poor’s; Bureau of Economic Analysis, Department of Commerce.
Comparative Statics: Changes in Interest Rate

- **Income effect:** if a saver $s > 0$, then higher interest rate increases income for given amount of saving. Increases consumption in first and second period. If borrower $s < 0$, then income effect negative.

- **Substitution effect:** gross interest rate $1 + r$ is relative price of consumption in period 1 to consumption in period 2. Current $c$ becomes more expensive relative to $c'$. This increases $c'$ and reduces $c$.

- **Hence:** for a saver an increase in $r$ increases $c'$ and may increase or decrease $c$. For a borrower an increase in $r$ reduces $c$ and may increase or decrease $c'$.
Figure 8.12 An Increase in the Real Interest Rate for a Lender
Figure 8.13  An Increase in the Real Interest Rate for a Borrower