Lecture 12
Dynamic General Equilibrium: Applications & Extensions

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Economics 312/702
Goods Market Equilibrium

- **Output supply**: increasing in real interest rate
  An increase in the interest rate raises current labor supply. This increases employment, raising output.

- **Output demand**: decreasing in real interest rate.
  Higher real interest rates reduce investment.
  Higher real interest rates reduce consumption.
Figure 9.18 The Complete Real Intertemporal Model

(a) \( w = \text{Current Real Wage} \)

\[ N^d \]

\[ N^s(r^*) \]

\( w^* \)

\( N^* \)

\( N = \text{Current Employment} \)

(b) \( r = \text{Real Interest Rate} \)

\[ y^d \]

\[ y^s \]

\( r^* \)

\( y^* \)

\( Y = \text{Current Output} \)
Increase in current TFP $z$

- Labor is more productive, increasing $N^d$. Output supply curve $Y^s(r)$ shifts out due to direct effect of $z$ and due to increase in labor input.
- No effect on output demand, since TFP unchanged in future, hence no change in $I^d(r)$
- In equilibrium, real interest rate falls, consumption and investment rise, employment rises, real wage rises.
- Productivity shocks are a potential explanation for business cycles, as we’ll discuss later.
- Ongoing productivity improvements are the main explanation for long run growth, as we discussed previously.
Figure 9.21  The Equilibrium Effects of an Increase in Current Total Factor Productivity
Increase in future TFP $z'$

- No (direct) effect on labor market (only through change in $r$), no effect on current production. Hence no effect on $Y^s(r)$.
- Investment demand $I^d(r)$ increases for any $r$, since $z'F_K(K', N')$ increases. Output demand curve shifts right.
- In equilibrium, real interest rate rises, investment increases, consumption may rise or fall, employment rises, real wage falls, output rises.
- That is, news of future productivity causes boom today. Important in explaining investment boom in the 1990s.
Figure 9.22 The Equilibrium Effects of an Increase in Future Total Factor Productivity
Figure 9.23  Percentage Deviations From Trend in GDP and Investment, 1990–2006
We now extend our two-period model to cover an infinite horizon economy. Ongoing savings, capital accumulation, production.

This will allow us to consider equilibrium growth dynamics, later add uncertainty and study business cycle fluctuations.

All of the features of the two-period model will still be present:

- Households trade off consumption and leisure within a period and over time.
- Firms hire labor each period until marginal product equal to wage, trade off forgone revenue of investment with increased productive capacity of more capital.

Each period markets for goods, labor, capital (savings/investment) will clear.
Household Problem: Infinite Horizon

- Add labor/leisure tradeoff to our previous infinite horizon consumption/savings problem.

Preferences:

$$\sum_{t=0}^{\infty} \beta^t u(C_t, 1 - N_t)$$

Flow budget constraint:

$$C_t + a_{t+1} = w_t N_t + x_t + (1 + r_t) a_t$$

As before, can write household problem with a single present value budget constraint:

$$\sum_{t=0}^{\infty} \frac{C_t}{\prod_{s=0}^{t}(1 + r_s)} = \sum_{t=0}^{\infty} \frac{x_t + w_t N_t}{\prod_{s=0}^{t}(1 + r_s)} + a_0 \equiv x^{PV}$$

Alternatively, can just include the flow constraint each period. We’ll do the latter now, like in Wickens.
Household Problem: Optimality Conditions

- Lagrangian:

\[ L = \sum_{t=0}^{\infty} \left[ \beta^t u(C_t, 1 - N_t) + \lambda_t (w_t N_t + x_t + (1 + r_t) a_t - C_t - a_{t+1}) \right] \]

- First order conditions for consumption \( C_t \), labor \( N_t \), assets \( a_{t+1} \):

\[
\beta^t u_C(C_t, 1 - N_t) = \lambda_t \\
\beta^t u_l(C_t, 1 - N_t) = \lambda_t w_t \\
\lambda_t = \lambda_{t+1}(1 + r_{t+1})
\]
Combining the first two gives the same intra-temporal optimality conditions for consumption/leisure:

$$\frac{u_l(C_t, 1 - N_t)}{u_C(C_t, 1 - N_t)} = w_t$$

Combining the first and third gives the Euler equation:

$$u_C(C_t, 1 - N_t) = \beta(1 + r_{t+1})u_C(C_{t+1}, 1 - N_{t+1})$$

As before, suppose that $r_t = r \ \forall t$, and $\beta(1 + r) = 1$. Euler equation then implies:

$$u_C(C_t, 1 - N_t) = u_C(C_{t+1}, 1 - N_{t+1})$$

Does not imply $C_t = C_{t+1}$ since $N_t$ could change, for example if $w_t \neq w_{t+1}$.
However if preferences are separable ($u_{Cl} = 0$):

$$u(C_t, 1 - N_t) = U(C_t) - V(N_t),$$

then:

$$u_C(C_t, 1 - N_t) = U'(c_t) \text{ independent of } N_t$$

Then Euler equation independent of $N_t$, so

$$\beta(1 + r) = 1 \Rightarrow C_t = C_{t+1}.$$

Even if preferences are non-separable $u_{Cl}(C_t, 1 - N_t) \neq 0$, if $w_t = w$ then

$$\frac{u_l(C_t, 1 - N_t)}{u_C(C_t, 1 - N_t)} = w$$

implies $N_t$ constant if $C_t$ is. So even then

$$\beta(1 + r) = 1 \Rightarrow C_t = C_{t+1}.$$

In either of these cases, with $C_t = C_{t+1}$ we have as before:

$$C_t = \frac{r}{1 + r} \sum_{s=0}^{\infty} \frac{w_{t+s}N_{t+s} + x_{t+s}}{(1 + r)^s} + r a_t$$
Firm Problem: Infinite Horizon

- Now extend firm problem to infinite horizon, with ongoing investment. Also allow firms to issue debt.
- Profits each period, with investment $I_t$, debt $b_t$:
  \[
  \pi_t = z_t F(K_t, N_t) - w_t N_t - I_t + b_{t+1} - (1 + r_t) b_t
  \]
- Capital evolution:
  \[
  K_{t+1} = (1 - \delta) K_t + I_t
  \]
- Substitute into profit function:
  \[
  \pi_t = z_t F(K_t, N_t) - w_t N_t - K_{t+1} + (1 - \delta) K_t + b_{t+1} - (1 + r_t) b_t
  \]
- Firm objective is to maximize discounted value of profits:
  \[
  \sum_{t=0}^{\infty} \frac{\pi_t}{\prod_{s=0}^{t}(1 + r_s)}
  \]
Firm Problem: Optimality Conditions

- **Firm objective:**

\[
\sum_{t=0}^{\infty} \left( z_t F(K_t, N_t) - w_t N_t - K_{t+1} + (1 - \delta) K_t + b_{t+1} - (1 + r_t) b_t \right) / \prod_{s=0}^{t}(1 + r_s)
\]

- **First order conditions for labor** \(N_t\), capital \(K_{t+1}\), debt \(b_{t+1}\):

\[
\frac{z_t F_N(K_t, N_t) - w_t}{\prod_{s=0}^{t}(1 + r_s)} = 0
\]

\[
\frac{1}{\prod_{s=0}^{t}(1 + r_s)} = \frac{z_{t+1} F_K(K_{t+1}, N_{t+1}) + 1 - \delta}{\prod_{s=0}^{t+1}(1 + r_s)}
\]

\[
\frac{1}{\prod_{s=0}^{t}(1 + r_s)} = \frac{1 + r_{t+1}}{\prod_{s=0}^{t+1}(1 + r_s)}
\]
Firm Problem: Optimality Conditions

- Labor optimality condition gives the same conditions we’ve seen before for labor demand:

\[ z_t F_N(K_t, N_t) = w_t \]

Firm hires labor until marginal product equal to real wage.

- Optimal choice of capital gives the same condition from the two period model:

\[ z_{t+1} F_K(K_{t+1}, N_{t+1}) - \delta = r_{t+1} \]

Firm invests until future marginal product net of depreciation equal to real interest rate.
Recall optimality condition for $b_{t+1}$:

$$\prod_{s=0}^{t}(1 + r_s) = \prod_{s=0}^{t+1}(1 + r_s)$$

This is an identity, or simply says $0 = 0$.

Optimal choice of debt is independent of $b_t$. That is, any debt sequence is optimal.

Debt is redundant: firm can finance investment either using profits (retained earnings) or by issuing debt. Indifferent between these choices.

In future, can ignore firm debt without loss of generality.
This is a simple example of **Modigliani-Miller (1958) theorem**: without distortions, the firm finance is irrelevant. Financing by equity, retained earnings, or debt all yield same firm value.

Modigliani summary: “With well-functioning markets (and neutral taxes) and rational investors, who can ‘undo’ the corporate financial structure by holding positive or negative amounts of debt, the market value of the firm – debt plus equity – depends only on the income stream generated by its assets. It follows, in particular, that the value of the firm should not be affected by the share of debt in its financial structure or by what will be done with the returns – paid out as dividends or reinvested (profitably).”
Miller analogy: “Think of the firm as a gigantic tub of whole milk. The farmer can sell the whole milk as it is. Or he can separate out the cream, and sell it at a considerably higher price than the whole milk would bring. If there were no costs of separation, the cream plus the skim milk would bring the same price as the whole milk.”

Increasing debt (cream) lowers the value of equity (skim milk) – selling off safe cash flows to debt-holders leaves the firm with more lower valued equity, keeping the total value of the firm unchanged. Any gain from using more of debt is offset by the higher cost of now riskier equity. Hence, given a fixed amount of total capital, the allocation of capital between debt and equity is irrelevant because the weighted average of the two costs of capital to the firm is the same for all possible combinations of the two.
Limitations of Modigliani-Miller

Like Ricardian equivalence, MM theorem is a strong result: firm capital structure does not matter. It relies on strong assumptions:
- No taxes
- No transaction costs
- No bankruptcy costs
- Equivalence in borrowing costs for both companies and investors
- Symmetry of market information
- No effect of debt on a company’s earnings before interest and taxes

Failure of any of these (as in practice) means MM capital structure will affect value of firm. Firms will have optimal capital structure.

Nonetheless, it is a useful starting point and a good approximation in many cases.
Recent Capital Structure Changes

- In recent years interest rates have been extremely low, allowing firms to issue debt cheaply.
- Many of these companies have used the proceeds from debt issuance to buy back their own shares. This has changed capital structure greatly: increasing debt, cutting equity. Market valuations of companies doing this have generally increased.
- Driven by tax reasons, reduced growth forecasts, market imperfections (increase earnings per share).
- Some concern that buybacks could lower future growth (lower investment), increase debt service burden when interest rates increase.
# Buyback Boom

Companies are spending billions buying back their own shares. For the first three quarters, this year’s buybacks have surpassed last year’s as the most since the 2008 financial crisis.

## Companies spending the most on buybacks through Sept. 30

<table>
<thead>
<tr>
<th>Company</th>
<th>Buybacks, in billions</th>
<th>Change in shares outstanding</th>
<th>Change in share price</th>
</tr>
</thead>
<tbody>
<tr>
<td>Apple</td>
<td>$30.2</td>
<td>▼2.8%</td>
<td>▲8.1%</td>
</tr>
<tr>
<td>Microsoft</td>
<td>14.2</td>
<td>▼3.0</td>
<td>▲16.7</td>
</tr>
<tr>
<td>Qualcomm</td>
<td>9.6</td>
<td>▼5.5</td>
<td>▼33.2</td>
</tr>
<tr>
<td>American International Group</td>
<td>7.5</td>
<td>▼7.6</td>
<td>▲11.1</td>
</tr>
<tr>
<td>Gilead Sciences</td>
<td>7.0</td>
<td>▼2.7</td>
<td>▲13.0</td>
</tr>
<tr>
<td>Oracle</td>
<td>6.8</td>
<td>▼2.9</td>
<td>▼12.5</td>
</tr>
<tr>
<td>Wells Fargo</td>
<td>6.7</td>
<td>▼1.0</td>
<td>▲1.8</td>
</tr>
<tr>
<td>AbbVie</td>
<td>6.3</td>
<td>▲3.9</td>
<td>▼6.6</td>
</tr>
<tr>
<td>Pfizer</td>
<td>6.2</td>
<td>▼2.1</td>
<td>▲3.3</td>
</tr>
<tr>
<td>Boeing</td>
<td>6.0</td>
<td>▼4.7</td>
<td>▲14.9</td>
</tr>
</tbody>
</table>

## Buybacks by all U.S. companies, through Sept. 30 of each year

$600 billion

*Through Nov. 20  †Data are incomplete  
Sources: Birinyi Associates; WSJ Market Data Group (share performance)

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So there may be deviations from MM, but we will take it as our starting point.
We now assume households own firms, so unearned income is firm profits $x_t = \pi_t$.

Now an equilibrium consists of an allocation, which is an infinite sequence $\{Y_t, K_t, N_t, C_t\}_{t=0}^{\infty}$, and factor prices at each date, $\{w_t, r_t\}_{t=0}^{\infty}$ such that:

- Given the prices $\{w_t, r_t\}_{t=0}^{\infty}$, the consumption and labor allocation solves the household problem. That is, Euler equation and optimality condition for labor hold.
- Given the prices $\{w_t, r_t\}_{t=0}^{\infty}$, the capital and labor allocation solves the firm problem. That is, optimality conditions for labor and capital hold.
The goods market clears:

\[ Y_t = z_t F(K_t, N_t) = C_t + I_t \]

The labor market clears: the same \( N_t \) solves the household and firm problems.

By Walras law, the capital market will clear as well. Household savings equals firm investment:

\[ a_t = K_t \]
Illustration of Walras Law

- If we assume that the production function is constant returns, then:

\[ Y_t = z_t F(K_t, N_t) = z_t F_K(K_t, N_t) + z_t F_N(K_t, N_t) \]

Then \( \pi_t = x_t = 0 \).

- Then evaluate the household budget constraint in equilibrium:

\[
C_t = w_t N_t + x_t + (1 + r_t) a_t - a_{t+1} \\
= z_t F_N(K_t, N_t) N_t + (1 + z_t F_K(K_t, N_t) - \delta) a_t - a_{t+1} \\
= z_t F(K_t, N_t) + (1 - \delta) a_t - a_{t+1} \\
= z_t F(K_t, N_t) + (1 - \delta) K_t - K_{t+1} \\
= Y_t - I_t
\]
From the household and firm optimality conditions for labor we have:

\[
\frac{u_l(C_t, 1 - N_t)}{u_C(C_t, 1 - N_t)} = z_t F_N(K_t, N_t)
\]

This is the same as in the static model, only now \( K_t \) will vary.

From the firm optimality condition for capital and the household Euler equation we have:

\[
u_C(C_t, 1 - N_t) = \beta u_C(C_{t+1}, 1 - N_{t+1})[z_{t+1} F_K(K_{t+1}, N_{t+1}) + 1 - \delta]\]

This is the same condition as in the optimal allocation from earlier in the class.

This is an illustration of the welfare theorems. The equilibrium allocation is the optimal allocation.
In fact, the two conditions that summarized the optimal allocation also characterize the equilibrium allocation when labor supply is inelastic. That is, if \( u_l = 0 \) so \( N_t = 1 \), then:

\[
\begin{align*}
    u_C(C_t, 0) &= \beta u_C(C_{t+1}, 0)[z_{t+1}F_K(K_{t+1}, 1) + 1 - \delta] \\
    z_t F(K_t, 1) &= C_t + K_{t+1} - (1 - \delta)K_t
\end{align*}
\]

The same phase diagrams we used to analyze the dynamics of the optimal allocation then also characterize equilibrium dynamics.

But when labor supply is elastic, we have to determine the joint dynamics of \( \{C_t, N_t, K_t\} \), not just \( \{C_t, K_t\} \).
Phase diagram: An reduction in labor from $N^*$ to $N'$. As before, initial effect depends on the slope of the saddle path.

$zF_K(K^*,N^*) = \delta + \theta$

$zF(K,N^*) - \delta K$

$zF_K(K',N')$

$zF(K,N') - \delta K$

$C^*$

$C_0$

$C'$

$K'$

$K^*$

$K$

Phase diagram: An reduction in labor from $N^*$ to $N'$. As before, initial effect depends on the slope of the saddle path.
• In steady state we have $C_t = C^*, K_t = K^*, N_t = N^*, z_t = z$:

\[ zF_K(K^*, N^*) = 1/\beta + \delta - 1 = \theta + \delta \]
\[ zF(K^*, N^*) = C^* + \delta K^* \]
\[ u_l(C^*, 1 - N^*) / u_C(C^*, 1 - N^*) = zF_N(K^*, N^*) \]

• Example:

\[ F(K, N) = K^\alpha N^{1-\alpha}, \ u(C, 1 - N) = \log C + \gamma \log(1 - N). \]

\[ z\alpha(K^*/N^*)^\alpha - 1 = \theta + \delta \]
\[ z(K^*)^\alpha(N^*)^{1-\alpha} = C^* + \delta K^* \]
\[ \gamma C^* \]
\[ 1 - N^* \]
\[ z(1 - \alpha)(K^*/N^*)^\alpha \]
Real Business Cycles

- We learned how to map preferences (for the household), technology (for the firm) and a government policy into a Competitive Equilibrium.

- If we let preferences, technology or the government preferences change over time, the equilibrium sequence will also fluctuate.

- We will use a model of this type to analyze business cycle fluctuations.

- All these (preferences, technology, policy) are real factors (as opposed to monetary).

- This is why we call this approach Real Business Cycles.

- Basic model: Brock and Mirman (1972)
We will have a shock: change in technology or policy.

Then we will have a propagation mechanism: intertemporal labor substitution and capital accumulation.

We will have fluctuations as an equilibrium outcome.

Main driving force is changes in productivity. Solow model emphasized TFP as source of growth. Now emphasize (random) variations in TFP as source of business cycles.

Basic idea: intertemporal substitution. When productivity is high, want to work more, produce more. When it is low, the reverse. Changes in productivity drive output.

Observation: TFP (Solow residual) and GDP are highly correlated.
Permanent Productivity Increase

- **Income effect**: higher consumption and higher leisure.
- **Substitution effect**: higher consumption and lower leisure.
- **Long-run facts suggest income and substitution effects cancel**.
  - Labor demand increases and labor supply falls.
  - Wages increase with no change in employment.
  - Output supply increases.
- **Consumption and investment demand increase**.
- **Long-run facts suggest no change in interest rates**.
Figure 9.21 The Equilibrium Effects of an Increase in Current Total Factor Productivity

(a) $w = \text{Current Real Wage}$

(b) $r = \text{Real Interest Rate}$

$N = \text{Current Employment}$

$Y = \text{Current Output}$

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Figure 9.22 The Equilibrium Effects of an Increase in Future Total Factor Productivity

(a) \( w = \text{Current Real Wage} \)

- \( N^d(r_1) \)
- \( N^s(r_2) \)
- \( w_1 \)
- \( w_2 \)
- \( N_1 \)
- \( N_2 \)

(b) \( r = \text{Real Interest Rate} \)

- \( y_1^d \)
- \( y_2^d \)
- \( y^s \)
- \( r_1 \)
- \( r_2 \)
- \( Y_1 \)
- \( Y_2 \)

\( N = \text{Current Employment} \)

\( Y = \text{Current Output} \)
Persistent Productivity Increase

- Productivity stays high for “a while” but comes back down.
- Substitution effect outweighs income effect in labor supply.
  - Labor demand increases more than labor supply falls.
  - Wages and employment increase.
  - Output supply increases
- Consumption and investment demand increase slightly.
- Output supply increases more than output demand.
- Interest rates fall and output increases.
Figure 11.3  Effects of a Persistent Increase in Total Factor Productivity the Real Business Cycle Model
Figure 11.3 Average Labor Productivity with Total Factor Productivity Shocks
# Table 11.1 Data Versus Predictions of the Real Business Cycle Model with Productivity Shocks

<table>
<thead>
<tr>
<th>Variable</th>
<th>Data</th>
<th>Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>Consumption</td>
<td>Pro-cyclical</td>
<td>Pro-cyclical</td>
</tr>
<tr>
<td>Investment</td>
<td>Pro-cyclical</td>
<td>Pro-cyclical</td>
</tr>
<tr>
<td>Price Level</td>
<td>Countercyclical</td>
<td>Countercyclical</td>
</tr>
<tr>
<td>Money Supply</td>
<td>Pro-cyclical</td>
<td>—</td>
</tr>
<tr>
<td>Employment</td>
<td>Pro-cyclical</td>
<td>Pro-cyclical</td>
</tr>
<tr>
<td>Real Wage</td>
<td>Pro-cyclical</td>
<td>Pro-cyclical</td>
</tr>
<tr>
<td>Average Labor Productivity</td>
<td>Pro-cyclical</td>
<td>Pro-cyclical</td>
</tr>
</tbody>
</table>