Midterm Examination Solutions

Instructions: This is a 75 minute examination worth 100 total points. Each question is worth 25 points. Choose FOUR out of the following FIVE questions. DO NOT ANSWER MORE THAN FOUR QUESTIONS. If you do, your grade will be based on the LOWEST four questions.

In order to get full credit, you must give a clear, concise, and correct answer, including all necessary calculations. Notes and books will not be permitted. Explain your answers clearly and use graphs when helpful.

1. Consider a government that must fund a given level of spending $G$, but does not have access to lump-sum taxes.

(a) The government imposes a proportional income tax $\tau$ on the representative consumer, so after-tax income is $(1 - \tau)(wN + rK + \pi)$. What effect does this have on the competitive equilibrium, compared to the case of lump-sum taxes?

Solution: Following our usual steps we can derive the equilibrium condition:

$$\frac{u_l(c, l)}{u_c(c, l)} = (1 - \tau)zF_N(K, N)$$

Note that as the tax $\tau$ drives a wedge between the marginal productivity of labor and the marginal rate of substitution, this equilibrium is not Pareto-optimal, since $MPN \neq MRS$. The tax lowers the effective real wage the household faces, which has income and substitution effects on labor supply which will at least partially offset. There is also a negative income effect due to the tax hitting non-labor income, which would increase labor supply. In the lump sum case, there is only the negative income effect. So with lump sum taxes, labor supply would increase, while with proportional taxes the effect will be smaller (labor supply will increase by less, could stay the same, or could possibly increase).

(b) The government instead imposes a consumption tax $t$, increasing the effective cost of consumption goods from 1 to $(1 + t)$. What effect does this have on the competitive equilibrium, compared to the case of lump-sum taxes?

Solution: The household problem for Robinson is now:

$$\max_{c,l} u(c, l)$$

$$s.t. \quad (1 + t)c = wN + rK + \pi$$
Note that we can rewrite the budget constraint as:

\[ c = \frac{1}{1 + t}(wN + rK + \pi) \]

Then following the usual steps we have:

\[ \frac{u(c, l)}{u_c(c, l)} = \frac{1}{1 + t}zF_N(K, N) \]

Thus this is just like the previous case with \( \frac{1}{1 + t} \) replacing \((1 - \tau)\). So the effects are the same as in the previous case. The tax drives a wedge between the marginal productivity of labor and the marginal rate of substitution, this equilibrium is not Pareto-optimal, since \( MPN \neq MRS \). The tax lowers the effective real wage the household faces, which has income and substitution effects on labor supply which will at least partially offset. There is also a negative income effect due to the tax hitting non-labor income, which would increase labor supply. In the lump sum case, there is only the negative income effect. So with lump sum taxes, labor supply would increase, while with proportional taxes the effect will be smaller (labor supply will increase by less, could stay the same, or could possibly increase).

(c) The government must decide whether to impose an income tax or a consumption tax. In either case, total tax receipts must equal \( G \). Does the representative consumer prefer one tax to the other? Explain.

**Solution:** As we’ve seen, the two taxes are equivalent with \( \frac{1}{1 + t} = (1 - \tau) \). Since they must raise the same revenue, the effects are the same. So the representative consumer is indifferent between them.

2. Consider the problem of how to eat a birthday cake. You are given a cake of size \( y \) on your birthday in the first period (and nothing on the next day). The only way to eat cake tomorrow is to save some of it, however part of the cake goes stale between today and tomorrow. So if you eat \( c \) in the first period, saving an amount \( s = y - c \) for tomorrow, when tomorrow comes you can only consume \( c' = (1 - x)s \), where \( x > 0 \) is the spoilage rate. You have standard preferences:

\[ u(c) + \beta u(c') \]

over your consumption of cake today \( (c) \) and tomorrow \( (c') \), and \( 0 < \beta < 1 \).

(a) Write down your decision problem and find the first order conditions for your optimal choice of cake consumption.

**Solution:** We can write the budget constraints as:

\[ c + s = y \]
\[ c' = (1 - x)s \]
So eliminating $s$ between the two equations gives the present value budget constraint:

$$c + \frac{c'}{1-x} = y.$$ 

Then the decision problem is:

$$\max_{c,c'} u(c) + \beta u(c') \quad \text{s.t.} \quad c + \frac{c'}{1-x} = y$$

We can form a Lagrangian with multiplier $\lambda$ on the constraint, which gives the first order conditions:

$$\text{FOC: } u'(c) = \lambda$$

$$\beta u'(c') = \frac{\lambda}{1-x}$$

If we combine them we get the Euler Equation:

$$u'(c) = \beta(1-x)u'(c')$$

(b) Do you decide to eat more cake today or tomorrow? 

**Solution:** This problem is like the standard savings problem but with a negative interest rate (due to the depreciation or spoilage of the cake). So since $\beta(1-x) < 1$, we have $u'(c) < u'(c')$ and so therefore $c > c'$. Thus it’s optimal to eat more cake today.

3. Suppose that instead of simply being a waste of output, government purchases are used for infrastructure which increases current productivity. In particular, suppose that the government increases its current purchases ($G$ increases) and that this in turn increases total factor productivity ($z$ increases). The government funds the higher expenditure by lump sum taxes. Using the static general equilibrium model from class (and diagrams when possible), discuss the effects this policy change will have on the equilibrium levels of output, employment, and consumption. Be sure to consider income and substitution effects.

**Solution:**

The increase in government spending in this example has two separate effects on the production possibilities frontier. First, the increase in government spending implies a parallel downward shift in the production possibilities frontier. Second, the productive nature of government spending is equivalent to an increase in total factor productivity that shifts the production possibilities frontier upward and increases its slope. If the production-enhancing aspects of the increase in government spending are large enough, representative consumer utility could rise.

There are three effects at work in this example. First, there is a negative income effect from the increase in taxes needed to pay for the increased government spending. This
effect tends to lower both consumption and leisure. Second, there is a substitution effect due to the productive effect of the increase in $G$. This effect tends to increase both consumption and leisure. Third, there is a positive income effect from the increase in $G$ on productivity. This effect tends to increase both consumption and leisure. In general, consumption may rise or fall, and leisure may rise or fall. The overall effect on output is the same as in any increase in total factor productivity. Output surely rises.

4. Suppose the representative household has preferences over consumption $c$ and leisure $l$ (with $h$ total hours in the day) given by:

$$u(c, l) = c - \frac{(h - l)^2}{2}.$$

The representative firm produces according to:

$$Y = zK^\alpha N^{1-\alpha}$$

where $z$ is total factor productivity and $K$ is a given fixed amount. Suppose that there is no government spending, so the household budget constraint is:

$$c = rK + wN.$$

The goods market clearing condition is thus:

$$Y = c$$

(a) Find the household’s labor supply function. How does it vary with the wage $w$?  
Solution: We can write the household Lagrangian:

$$L = c - \frac{N^2}{2} - \lambda \{c - rK - wN\}$$

Taking first order conditions gives

$$1 - \lambda = 0$$
$$N - \lambda w = 0.$$ 

Labor supply is thus given by:

$$N = w. \quad (1)$$ 

Thus labor supply is increasing in the wage, and in particular moves one-for-one with the wage.

(b) Solve for the equilibrium levels of output, labor, and consumption.  
Solution: We have to solve for $Y$, $N$, and $c$ as functions of exogenous variables \{z, K\}. The firm’s optimality conditions are given by:

$$w = (1 - \alpha)zK^\alpha N^{-\alpha} \quad \text{(2)}$$
$$r = \alpha zK^{\alpha-1}N^{1-\alpha} \quad \text{(3)}$$
Combining these with the household conditions gives:

\[
N = [(1 - \alpha)zK^\alpha]\frac{1}{1+\alpha}
\]
\[
Y = zK^\alpha N^{1-\alpha}
\]
\[
c = Y
\]
\[
= zK^\alpha [(1 - \alpha)zK^\alpha]\frac{1}{1+\alpha}.
\]

5. Consider a two period model in which the government runs a loan program. Loans are made to consumers in the first period of the model at the market interest rate \(r\), with the total quantity of loans denoted \(L\). The government loans are financed via lump sum taxes on consumers in the first period, and government spending is zero in both periods. In the second period, when the loans are repaid, the government rebates this amount as lump-sum transfers to consumers.

(a) Write down the budget constraints of the government in the first and second periods, and then derive its present value budget constraint.

**Solution:** There is no government spending in either period. In the first period, the government must collect lump-sum taxes so that \(T = L\). In the second period, a lump-sum rebate is given so that \(T' = -(1 + r)L\). The present-value government budget is therefore:

\[
T + \frac{T'}{1 + r} = 0
\]

(b) What is the effect of this program on the consumption and savings decisions of households? Explain your answer.

**Solution:** The consumers’ budget constraints can be written as follows:

\[
c + s = y + L - T
\]
\[
c' = (1 + r)s + y' - T' - (1 + r)L
\]

where \(L\) represents the size of the loan that the individual consumer takes from the loan program. Combining, we obtain:

\[
c + \frac{c'}{1 + r} = y - T + \frac{y' - T'}{1 + r}.
\]

The government loan program is a perfect substitute for private borrowing, as both earn the same interest rate \(r\). The size, \(L\), of the loan program, does not change the fact that the present value of tax collections must equal zero. So the loan program has no effect on the consumption decisions of households. It will affect private saving as all that really matters for the household is \(s - L\) (which we can see from the budget constraint). So an increase in \(L\) is offset by an increase in saving \(s\), leaving \(s - L\) unchanged.