Midterm Examination Solutions

FOR GRADUATE STUDENTS ONLY

Instructions: This is a 75 minute examination worth 100 total points. Question 1 is worth 40 points, all other questions are worth 30 points. ANSWER QUESTION 1 then choose TWO of following THREE questions. DO NOT ANSWER ALL OF THE QUESTIONS. If you do, your grade will be based on the LOWEST of the questions.

In order to get full credit, you must give a clear, concise, and correct answer, including all necessary calculations. Notes and books will not be permitted. Explain your answers clearly and use graphs when helpful.

ANSWER THIS QUESTION (40 points)

1. Consider a variation on the Solow model where the savings rate is variable instead of constant. In particular suppose that as usual output is produced competitively via a Cobb-Douglas production function: \( Y = K^\alpha N^{1-\alpha} \), and the population grows at the constant rate \( n > 0 \). Suppose that there is no depreciation or productivity growth (\( \delta = g = 0 \)), and that total savings is given by \( S_t = s(r)Y_t \), where:

\[
s(r) = \bar{s}r^\phi
\]

where we introduce the constants \( \bar{s} > 0 \) and \( \phi > 0 \). Thus \( s(r) \) is increasing in \( r \), so that higher real interest rates induce higher savings rates.

(a) Solve for the steady state equilibrium per-worker quantities of capital, output, and consumption (as a function of \( \alpha, \phi, \bar{s}, n \)) if \( \phi < \frac{\alpha}{1-\alpha} \).

Solution: As usual, we introduce the notation \( k = K/N, y = Y/N = k^\alpha \). Then

\[
\dot{K} = s(r)Y
\]

\[
\dot{k} = \frac{\dot{K}}{N} - \frac{K \dot{N}}{NN} = s(r)k^\alpha - nk = \bar{s}r^\phi k^\alpha - nk
\]

From firm’s optimization problem

\[
r = MPK = \alpha k^{\alpha-1}
\]

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Thus the fundamental equation of the Solow Model is
\[ \dot{k} = \bar{s} \alpha^\phi k^{\alpha + \phi(\alpha - 1)} - nk \]
The steady state is at \( \dot{k} = 0 \), where
\[ k^* = \left( \frac{\bar{s} \alpha^\phi}{n} \right) \left( \frac{1}{1 + \phi(1 - \alpha)} \right) \]
\[ y^* = (k^*)^\alpha = \left( \frac{\bar{s} \alpha^\phi}{n} \right) \left( \frac{1}{1 + \phi(1 - \alpha)} \right) \]
\[ c^* = (1 - s(r))y^* = \left[ 1 - n \left( \frac{\bar{s} \alpha^\phi}{n} \right) \right] \left( \frac{\bar{s} \alpha^\phi}{n} \right) \left( \frac{1}{1 + \phi(1 - \alpha)} \right) \]

(b) What are the effects of an increase in the interest elasticity of savings \( \phi \) on the per-worker quantities of capital, output, and consumption? Consider both the short-run and long-run effects, and continue to assume \( \phi < \frac{\alpha}{1 - \alpha} \) even after the increase.

**Solution:** First consider the long-run effects of an increase in \( \phi \) on \( k^* \), \( y^* \), and \( c^* \), we just need to check the sign of \( \frac{\partial k^*}{\partial \phi} \), \( \frac{\partial y^*}{\partial \phi} \), \( \frac{\partial c^*}{\partial \phi} \).

Define \( u(\phi) = \frac{\bar{s} \alpha^\phi}{n} \), \( v(\phi) = \frac{1}{(1 + \phi)(1 - \alpha)} \), then
\[ k^* = u(\phi)^v(\phi) = e^{v(\phi)\ln u(\phi)} \]
Thus
\[ \frac{\partial k^*}{\partial \phi} = (u(\phi)^v(\phi))' = (e^{v(\phi)\ln u(\phi)})' = e^{v(\phi)\ln u(\phi)} (v(\phi)\ln u(\phi))' = u(\phi)^v(\phi) \left( v'(\phi)\ln u(\phi) + v(\phi)\frac{u'(\phi)}{u(\phi)} \right) \]

Since \( u'(\phi) = \frac{\bar{s} \alpha^\phi \ln(n \alpha)}{n} \), \( v'(\phi) = \frac{-1}{(1 - \alpha)(1 + \phi)^2} \), we have
\[ \frac{\partial k^*}{\partial \phi} = \left( \frac{\bar{s} \alpha^\phi}{n} \right) \left( \frac{1}{1 + \phi(1 - \alpha)} \right) \left( \frac{-1}{(1 - \alpha)(1 + \phi)^2} \ln \left( \frac{\bar{s} \alpha^\phi}{n} \right) + \frac{1}{(1 + \phi)(1 - \alpha) \ln(1 - \alpha)} \right) \]

If \( \ln \alpha > \frac{1}{(1 + \phi)} \ln \left( \frac{\bar{s} \alpha^\phi}{n} \right) \), \( \frac{\partial k^*}{\partial \phi} > 0 \), \( k^*_{\text{NEW}} > k^* \), \( y^*_{\text{NEW}} > y^* \);
if \( \ln \alpha < \frac{1}{(1 + \phi)} \ln \left( \frac{\bar{s} \alpha^\phi}{n} \right) \), \( \frac{\partial k^*}{\partial \phi} < 0 \), \( k^*_{\text{NEW}} < k^* \), \( y^*_{\text{NEW}} < y^* \).
Similarly,

\[
\frac{\partial c^*}{\partial \phi} = \alpha \left( \frac{\bar{s} \alpha^\phi}{n} \right)^{\frac{1}{1+\phi}(1-\alpha)} \left( \frac{-1}{(1-\alpha)(1+\phi)} \ln \frac{\bar{s} \alpha^\phi}{n} + \frac{1}{(1+\phi)(1-\alpha)} \ln \alpha \right) 
\]

\[-n \left( \frac{\bar{s} \alpha^\phi}{n} \right)^{\frac{1}{1+\phi}(1-\alpha)} \left( \frac{-1}{(1-\alpha)(1+\phi)} \ln \frac{\bar{s} \alpha^\phi}{n} + \frac{1}{(1+\phi)(1-\alpha)} \ln \alpha \right) 
\]

\[= \left( \frac{-1}{(1-\alpha)(1+\phi)} \ln \frac{\bar{s} \alpha^\phi}{n} + \frac{1}{(1+\phi)(1-\alpha)} \ln \alpha \right) \left( \alpha \left( \frac{\bar{s} \alpha^\phi}{n} \right)^{\frac{1}{1+\phi}(1-\alpha)} - n \left( \frac{\bar{s} \alpha^\phi}{n} \right)^{\frac{1}{1+\phi}(1-\alpha)} \right) 
\]

\[= \frac{1}{(1+\phi)(1-\alpha)} \left( \frac{-1}{(1+\phi)} \ln \frac{\bar{s} \alpha^\phi}{n} + \ln \alpha \right) \left( \frac{\bar{s} \alpha^\phi}{n} \right)^{\frac{1}{1+\phi}(1-\alpha)} \left( \alpha - n \left( \frac{\bar{s} \alpha^\phi}{n} \right)^{\frac{1}{1+\phi}(1-\alpha)} \right) 
\]

If \(\frac{-1}{(1+\phi)} \ln \frac{\bar{s} \alpha^\phi}{n} + \ln \alpha \left( \alpha - n \left( \frac{\bar{s} \alpha^\phi}{n} \right)^{\frac{1}{1+\phi}(1-\alpha)} \right) > 0\), \(\frac{\partial c^*}{\partial \phi} > 0\), \(c^*_\text{NEW} > c^*\);

if \(\frac{-1}{(1+\phi)} \ln \frac{\bar{s} \alpha^\phi}{n} + \ln \alpha \left( \alpha - n \left( \frac{\bar{s} \alpha^\phi}{n} \right)^{\frac{1}{1+\phi}(1-\alpha)} \right) < 0\), \(\frac{\partial c^*}{\partial \phi} < 0\), \(c^*_\text{NEW} < c^*\).

As for the short run, capital, output and consumption will converge to the new steady state.

We can also draw the graph as follows. Suppose \(0 < \alpha < 1\), and \(g(\phi) = s(r)y = \bar{s} \alpha^\phi k^{\alpha+\phi(\alpha-1)}\) is the actual investment, \(nk\) is the break-even investment, then \(g'(\phi) = \bar{s} \alpha^\phi k^{\alpha+\phi(\alpha-1)} \ln \alpha k^{\alpha-1}\).

When \(\alpha k^{\alpha-1} > 1 \iff k < \alpha \frac{1}{1-\alpha}\), \(g'(\phi) > 0\), the new actual investment lies above the old one;

When \(\alpha k^{\alpha-1} < 1 \iff k > \alpha \frac{1}{1-\alpha}\), \(g'(\phi) < 0\), the new actual investment lies below the old one.

(c) Now suppose \(\phi = \frac{\alpha}{1-\alpha}\). How does this change the model? What are the effects of an increase in the saving fraction \(\bar{s}\) on capital and output now?
Solution: If $\phi = \frac{\alpha}{1-\alpha}$, then
$$\dot{k} = \bar{s}\alpha^{\frac{\alpha}{1-\alpha}} - nk$$

The steady state is at $\dot{k} = 0$, where
$$k^* = \frac{\bar{s}\alpha^{\frac{\alpha}{1-\alpha}}}{n}$$
$$y^* = (k^*)^\alpha = \left(\frac{\bar{s}\alpha^{\frac{\alpha}{1-\alpha}}}{n}\right)^\alpha$$

An increase in $\bar{s}$ increases $k^*$ and $y^*$ in the long run, and in the short run, capital and output will converge to the new steady state.

ANSWER TWO OF THE FOLLOWING THREE QUESTIONS (30 points each)

2. Consider the optimal growth model with inelastic labor supply, and for simplicity assume that there is no population or productivity growth. Household preferences are:
$$\sum_{t=0}^{\infty} \beta^t u(C_t)$$

The capital evolution equation is:
$$K_{t+1} = (1 - \delta) K_t + F(K_t) - C_t.$$  
Suppose the economy is initially in the steady state and then there is an unexpected and permanent increase in household patience, so $\beta$ increases (or $\theta = 1/\beta - 1$ falls).

(a) What are the long-run effects of this change on consumption and capital?

Solution: The Lagrangian function is
$$\mathcal{L} = \sum_{t=0}^{\infty} \left(\beta^t u(C_t) + \lambda_t [(1 - \delta) K_t + F(K_t) - C_t - K_{t+1}]\right)$$

By solving the FOCs with respect to $C_t$ and $K_{t+1}$, we can get the Euler Equation:
$$u'(C_t) = \beta u'(C_{t+1})(1 + F'(K_{t+1}) - \delta)$$

where $\beta = \frac{1}{1+\theta}$.

In Steady State, $C_t = C_{t+1}$, from the Euler Equation, we get the $\Delta C = 0$ curve
$$F'(K_{t+1}) = \delta + \theta$$

In Steady State, $K_t = K_{t+1}$, from the capital evolution equation, we get the $\Delta K = 0$ curve
When $\theta = 1/\beta - 1$ falls, the $\Delta K = 0$ curve is unaffected, while the $\Delta C = 0$ curve shift to the right. Thus the $C^*$ and $K^*$ both increase in the long run. We can draw the graph as follows:

(b) What happens to consumption and capital at the time of the change?

**Solution:** At the time of the change, $C$ jumps down to the saddle path, while $K$ is unchanged at that time. After that, both $C$ and $K$ move along the saddle path to the new steady state.

3. Consider a two period problem where a consumer has preferences over consumption in the two periods given by:

$$\log c + \beta \log c'.$$

She has no initial assets and has income $y$ in the first period, $y'$ in the second, pays taxes (net of benefits) $T$ in the first and $T'$ in the second, and can borrow and lend ($s$) at interest rate $r$, thus giving the two flow constraints:

$$c + s = y - T, \quad c' = (1 + r)s + y' - T'.$$

The government finances spending through taxes and borrowing:

$$G = T + B, \quad G' + (1 + r^G)B = T',$$

where the government borrows at a lower rate than households: $r^G < r$.

(a) Solve for the agent’s optimal consumption choices $c$ and $c'$ and savings $s$. 

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Solution: The present value budget constraint is

\[ c + \frac{c'}{1 + r} = y - T + \frac{y' - T'}{1 + r} \equiv y^{PV} \]

The Lagrangian function is

\[ \mathcal{L} = \log c + \beta \log c' + \lambda \left( y^{PV} - c - \frac{c'}{1 + r} \right) \]

FOCs:

\[ \frac{1}{c} = \lambda \]
\[ \frac{\beta}{c'} = \frac{\lambda}{1 + r} \]

We can get the consumption Euler equation

\[ \frac{1}{c} = \beta (1 + r) \frac{1}{c'} \]

Combined with the present value life time budget constraint, we have

\[ c = \frac{1}{1 + \beta} \left( y - T + \frac{y' - T'}{1 + r} \right) \]
\[ c' = \frac{\beta (1 + r)}{1 + \beta} \left( y - T + \frac{y' - T'}{1 + r} \right) \]
\[ s = \frac{\beta}{1 + \beta} (y - T) - \frac{1}{1 + \beta} \left( \frac{y' - T'}{1 + r} \right) \]

(b) Now suppose that the government cuts taxes in the current period, so \( T \) falls by some amount \( \Delta \), but government spending is unchanged. Thus future taxes must rise to pay back the principal and interest on the deficit this policy creates. How does this affect the consumer’s optimal choices?

Solution: The government present value budget constraint is

\[ T + \frac{T'}{1 + r^G} = G + \frac{G'}{1 + r^G} \]

The new tax is \( \tilde{T} = T - \Delta \), while government spending is unchanged. So

\[ \tilde{T'} = T' + (1 + r^G) \Delta \]

The new life time income for the consumer is

\[ \tilde{y}^{PV} = \frac{y - \tilde{T} + \frac{y' - \tilde{T}'}{1 + r}}{1 + r} \]
\[ = \frac{y - T + \frac{y' - T'}{1 + r}}{1 + r} + \Delta - \frac{1 + r^G}{1 + r} \Delta \]
\[ = y^{PV} + \frac{r - r^G}{1 + r} \Delta \]
Since \( r^G < r \), we have \( \tilde{y}^{PV} > y^{PV} \). Thus the consumer will increase consumption in both periods.

4. Consider the two-period dynamic general equilibrium model, which we can depict graphically as in class with equilibrium in the labor market (labor supply and demand) and the goods market (output supply and demand). Suppose the economy is initially in equilibrium, and then a new government program is announced. This program will make public infrastructure investments in the current period that will be funded by lump sum tax revenue and will increase future productivity. That is, the program combines an increase in \( G \) today (only, not \( G' \) as well) with an increase in \( z' \) in the future. As in class, assume that the response of labor supply to interest rates is small.

(a) What effect will the program have on consumption and investment demand, and thus on output demand?

**Solution:** The program combines an increase in \( G \) today with an increase in \( z' \) in the future. **Distinguish between shift of the demand, supply curve and change of the equilibrium result.**

If we just consider an increase in \( G \) today, then increase in current or future taxes reduces household wealth, thus consumption demand \( C^d \) shifts to the left, but by less than the amount \( G \) shifts to the right, and \( Y^d(r) = C^d(r) + I^d(r) + G \) shifts to the right.

If we just consider an increase in \( z' \) in the future, since \( z'F_K(K, N) \) increases, \( I^d(r) \) shifts to the right, \( Y^d(r) = C^d(r) + I^d(r) + G \) shifts to the right.

In sum, \( C^d \) shifts to the left, but by less than the amount \( G \) shifts to the right, \( I^d(r) \) shifts to the right, and \( Y^d(r) = C^d(r) + I^d(r) + G \) shifts to the right.

(b) What effect will this program have on labor supply and labor demand? How will the program affect the output supply curve?

**Solution:**

If we just consider an increase in \( G \) today, then increase in current or future taxes reduces household wealth, thus leisure falls and so labor supply \( N^s \) increases (shifts to the right), and output supply \( Y^s(r) \) increases (shifts to the right). There is no effect on labor demand curve \( N^d \).

If we just consider an increase in \( z' \) in the future, there is no (direct) effect on labor market (only through change in \( r \), if \( r \) increases, \( N^s \) shifts to the right), hence no (direct) effect on \( Y^s(r) \). There is no effect on labor demand curve \( N^d \).

In sum, \( N^s \) shifts to the right, there is no effect on labor demand curve \( N^d \), \( Y^s(r) \) shifts to the right.

(c) What will be the net equilibrium effects on output, interest rates, employment, and wages?

**Solution:** We can draw the equilibrium effects in the two graphs below. The equilibrium output increases, interest rate increases, employment increases, wages decrease.