An Example

- Now work out a parametric example, using standard functional forms. Cobb-Douglas production:

\[ y = A k^\alpha \]

- For preferences, set:

\[ U(c) = \frac{c^{1-\sigma} - 1}{1 - \sigma} \]

For \( \sigma > 0 \). Interpret \( \sigma = 1 \) as \( U(c) = \log c \).

- These imply the Euler equation:

\[ c_t^{-\sigma} = \beta c_{t+1}^{-\sigma} \left[ 1 + \alpha A k_{t+1}^{\alpha - 1} - \delta \right] = \beta c_{t+1}^{-\sigma} R_{t+1} \]

- For these preferences \( \sigma \) gives the curvature and so governs how the household trades off consumption over time.
Recall the Euler equation:

\[ c_t^{-\sigma} = \beta c_{t+1}^{-\sigma} [1 + \alpha A k_{t+1}^{\alpha-1} - \delta] \]

Steady state:

\[ F'(k^*) = A \alpha (k^*)^{\alpha-1} = \delta + \theta \]

\[ \Rightarrow k^* = \left( \frac{\alpha A}{\delta + \theta} \right)^{\frac{1}{1-\alpha}} \]

Then we get consumption:

\[ c^* = A (k^*)^\alpha - \delta k^* \]

\[ = A \left( \frac{\alpha A}{\delta + \theta} \right)^{\frac{\alpha}{1-\alpha}} - \delta \left( \frac{\alpha A}{\delta + \theta} \right)^{\frac{1}{1-\alpha}} \]
Steady state capital stock determined by:

\[ F'(k^*) = \delta + \theta \]

Consumption \( c^* \) increasing in \( k^* \) (since below golden rule level).

- If \( \delta \uparrow \), then \( k^* \downarrow \) so \( c^* \downarrow \).
- If total factor productivity \( \uparrow \) then \( k^* \uparrow \), so \( c^* \uparrow \).
An increase in the depreciation rate
An increase in total factor productivity

\[ \delta k \]

\[ \delta + \theta \]

\[ A'F(k) \]

\[ AF(k) \]

\[ c' + \delta k' \]

\[ c^* + \delta k^* \]

\[ k^* \]

\[ k' \]
Last time we analyzed the basic model of optimal growth and studied the behavior of the model in the steady state.

Now will analyze the dynamics of the model.

The model is nonlinear and in general does not allow for a complete, explicit solution in general.

However we can analyze the qualitative dynamics by analyzing a phase diagram summarizing the key equations of the model.

This will allow us to analyze both the short-run and long-run effects of changes in exogenous variables. For example, to see how the economy responds to a change in technology.
We will analyze the joint dynamics of \( \{c_t, k_t\} \). In any period \( t \), \( k_t \) is given and \( c_t \) is chosen optimally (as is \( k_{t+1} \)).

The key equations of the model are:

\[
U'(c_t) = \beta U'(c_{t+1})[F'(k_{t+1}) + 1 - \delta]
\]

\[
k_{t+1} = (1 - \delta)k_t + F(k_t) - c_t
\]

We’ll use the first to determine the dynamics of \( c \), the second the dynamics of \( k \).

In steady state, \( \Delta c_{t+1} = c_{t+1} - c_t = 0 \), and

\[
F'(k^*) = \delta + \theta
\]

If \( k < k^* \), then \( F'(k) > F'(k^*) \), so to satisfy Euler equation we need \( U'(c_{t+1}) < U'(c_t) \) and so \( c_{t+1} > c_t \). Similarly if \( k > k^* \), \( \Delta c < 0 \).
\[ \Delta c = 0: F'(k^*) = \delta + \theta \]

Dynamics of consumption

\[ \Delta c > 0 \quad \Delta c < 0 \]

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A key equation of the model is:

\[ k_{t+1} = (1 - \delta)k_t + F(k_t) - c_t \]

In steady state, \( \Delta k_{t+1} = k_{t+1} - k_t = 0 \), and

\[ c = F(k) - \delta k \]

If \( c_t < F(k_t) - \delta k_t \) then \( i_t > dk_t \), so \( \Delta k_{t+1} > 0 \). Similarly if \( c_t > F(k_t) - \delta k_t \) then \( \Delta k_{t+1} < 0 \).

Putting together the dynamics of \( c \) with the dynamics of \( k \) gives the phase diagram.
Dynamics of capital

\[ \Delta k = 0: F(k) - \delta k \]

\[ \Delta k > 0 \]

\[ \Delta k < 0 \]
Δc=0: F'(k*)=δ+θ

Δk=0: F(k)-δk

Phase diagram: dynamics of consumption and capital
From the phase diagram we can see the dynamics of \( \{k_t, c_t\} \) from any initial \((k_0, c_0)\).

But given \(k_0\) the dynamics from an arbitrary \(c_0\) will not converge to the steady state.

In general either \(c_t\) or \(k_t\) will go to zero. These are not part of an optimal solution.

However given \(k_0\) there is a unique value of \(c_0\) such that the economy converges to the steady state. This is the **saddle path**.

The optimal solution will be on the saddle path, as \(c_0\) is a function of \(k_0\) and will be chosen so that the economy is stable and converges to the steady state.
Phase diagram: saddle path and dynamics from different initial consumption levels

\[ \Delta c = 0: F'(k^*) = \delta + \theta \]

\[ \Delta k = 0: F(k) - \delta k\]
With the phase diagram, we can determine how an exogenous change affects $c_t$ and $k_t$ both over time and in the long run.

Suppose, for example, that households became less patient, so $\theta$ increases and $\beta$ falls. What would happen immediately and in the long run?

Recall the dynamics:

$$\Delta c = 0 : F'(k^*) = \delta + \theta$$

this curve shifts to the left, so steady state $k^*$ would fall

And for capital,

$$\Delta k = 0 : c = F(k) - \delta k$$

this curve is unaffected.

In the long run, $k_t$ and $c_t$ will fall. When the change in $\theta$ happens $c_t$ will jump up to the new saddle path.
Phase diagram: An increase in the discount rate ($\theta$).

When households become more impatient, they increase consumption, and save less. In the short run this leads to more consumption. But in the long run, the lower investment will lead to a reduction in capital and hence consumption.
• If total factor productivity increases, we already have seen that in steady state $k^* \uparrow$, so $c^* \uparrow$. But what happens along the transition?

• Recall the dynamics:

$$\Delta c = 0 : AF'(k^*) = \delta + \theta$$

this curve shifts to the right

• And for capital,

$$\Delta k = 0 : c = AF(k) - \delta k$$

this curve shifts up

• In the long run $(k^*, c^*)$ increase. But the effect in the short run depends on the slope of the saddle path, which in turn depends on how willing the household is to substitute over time.
If the household is relatively **impatient** (low $\beta$, high $\theta$), and is **unwilling** to substitute over time (low IES), then it will want to smooth consumption over time and value early periods highly. Then $c_t$ will increase and the economy will slowly move to the steady state. $c_0$ will increase.

If the household is relatively **patient** (high $\beta$, low $\theta$), and is **willing** to substitute over time (high IES), then it will forgo current consumption, invest more, and get to the steady state more quickly. $c_0$ may fall.

In both cases, $k_t$ increases each period until it reaches the steady state. In the long run $c_t$ increases, but in the short-run it may increase or decrease.
Phase diagram: An increase in total factor productivity ($A$), with a flatter saddle path (low IES, high discount rate)

$$A'F'(k^*) = \delta + \theta$$

$$A'F(k) - \delta k$$

Phase diagram: An increase in total factor productivity ($A$), with a flatter saddle path (low IES, high discount rate)
Phase diagram: An increase in total factor productivity (A), with a steep saddle path (high IES, low discount rate).

\[ AF'(k^*) = \delta + \theta \]

\[ A'F'(k^*) = \delta + \theta \]

\[ A'F(k) - \delta k \]

\[ c^* \]

\[ c' \]

\[ c0 \]

\[ k^* \]

\[ k' \]
There is one known case where we can work out an explicit solution.

Set $\delta = 1$ (full depreciation) use logarithmic utility, Cobb-Douglas:

$$U(c) = \log c, \quad F(k) = Ak^\alpha$$

Specialize the key equilibrium equations:

$$\frac{1}{c_t} = \frac{\beta \alpha Ak_t^{\alpha-1}}{c_{t+1}}$$
$$c_t = Ak_t^\alpha - k_{t+1}$$
• Guess that the solution is a constant savings rate $s$:

$$c_t = (1 - s)y_t$$

• Substitute into conditions:

$$\frac{1}{(1 - s)Ak_t^\alpha} = \frac{\beta\alpha Ak_{t+1}^{\alpha-1}}{(1 - s)Ak_{t+1}^\alpha} = \frac{\beta\alpha}{(1 - s)k_{t+1}} = \frac{\beta\alpha}{(1 - s)sAk_t^\alpha}$$

• So $s = \beta\alpha$, and do $c_t = (1 - \beta\alpha)Ak_t^\alpha$. 
In this special case we have the explicit relationship between \((c, k)\), the optimal decision rule or the saddle path.

We then have the dynamics of \(k_t\):

\[
k_{t+1} = sA k_t^\alpha
\]

The steady state is a special case of what we had earlier \((\delta = 1)\):

\[
k^* = \left( \frac{\alpha A}{1 + \theta} \right)^{\frac{1}{1-\alpha}}
\]

So we can now trace out the dynamics explicitly. For example, if \(A\) increases, \(c_t\) increases on impact and grows over time to the new steady state.