Lecture 14
More on Real Business Cycles

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Economics 312
Optimality Conditions

- Euler equation under uncertainty:
  \[ u_C(C_t, 1-N_t) = \beta E_t [u_C(C_{t+1}, 1-N_{t+1}) (1 + z_{t+1} F_K(K_{t+1}, N_{t+1}) - \delta)] \]

- Labor market optimality:
  \[ \frac{u_l(C_t, 1-N_t)}{u_C(C_t, 1-N_t)} = z_tF_N(K_t, N_t) \]

- Goods market clearing:
  \[ K_{t+1} = z_tF(K_t, N_t) + (1 - \delta)K_t - C_t \]
Evolution of the Technology

- $z_t$ changes randomly over time. Ignore growth and just think of fluctuations around a trend.
- We assume it follows the process:

\[
\begin{align*}
    \log z_t &= \rho \log z_{t-1} + \varepsilon_t \\
    \varepsilon_t &\sim \mathcal{N}(0, \sigma^2)
\end{align*}
\]

- This process is called $AR(1)$: an autoregression of order 1.
- The parameter $\rho$ governs how persistent are the changes in TFP. If $\rho = 1$ they are permanent. If $0 < \rho < 1$ they are persistent but eventually die out.
Examples of TFP Processes
This economy has a unique competitive equilibrium.

This economy satisfies the conditions that assure that both welfare theorems hold.

Why is this important?

Practical: We can solve instead the Social Planner’s Problem associated with it.

Normative: Business cycles in the model are efficient.

Fluctuations are the optimal response to a changing environment. They are not sufficient for inefficiencies or for government intervention. In this model the government can only worsen the allocation.
The previous problem does not have a known “paper and pencil” analytic solution.

Analysis of the model requires some approximations (such as linearization) or numerical analysis.

One of the main tools of modern macroeconomic theory is the computer.

We build small laboratory economies inside our computers and we run experiments on them. These give predictions about the real economy.

Modern macroeconomics is quantitative
Solving the Model in a Special Case

- There is one known case where we can work out an explicit solution.
- Set $\delta = 1$ (full depreciation) use our Cobb-Douglas production, and log utility:
  \[ u(C, 1 - N) = (1 - a) \log C + a \log(1 - N). \]
- Specialize the key equilibrium conditions:
  \[
  \frac{aC_t}{(1-a)(1-N_t)} = (1-\alpha) z_t K_t^\alpha N_t^{-\alpha}
  \]
  \[
  \frac{1}{C_t} = \beta E \left[ \frac{\alpha z_{t+1} K_{t+1}^{\alpha-1} N_{t+1}^{1-\alpha}}{C_{t+1}} \right]
  \]
  \[
  K_{t+1} = z_t K_t^\alpha N_t^{1-\alpha} - C_t
  \]
Make the following guesses:

\[ C_t = (1 - s) Y_t, \quad N_t = \bar{N} \]

Constant saving rate \( s \), constant labor supply \( \bar{N} \).

Substitute into conditions:

\[
\frac{a(1 - s) z_t K_t^\alpha \bar{N}^{1-\alpha}}{(1 - a)(1 - \bar{N})} = (1 - \alpha) z_t K_t^\alpha \bar{N}^{-\alpha}.
\]

\[
\frac{1}{(1 - s) z_t K_t^\alpha \bar{N}^{1-\alpha}} = \beta E \left[ \frac{\alpha z_{t+1} K_{t+1}^{\alpha-1} \bar{N}^{1-\alpha}}{(1 - s) z_{t+1} K_{t+1}^{\alpha-1} \bar{N}^{1-\alpha}} \right]
\]

\[
= \beta E \left[ \frac{\alpha}{(1 - s) K_{t+1}} \right]
\]

\[
= \beta E \left[ \frac{\alpha}{(1 - s) s z_t K_t^\alpha \bar{N}^{1-\alpha}} \right]
\]

\[ \Rightarrow s = \beta \alpha \]
This special case is then similar to the Solow model: constant savings rate. Constant labor supply (no growth). Difference is random shocks.

Now $K_{t+1} = sY_t$, so

$$
Y_{t+1} = z_{t+1} K_{t+1}^\alpha \bar{N}^{1-\alpha} \\
= z_{t+1} (sK_t)^\alpha \bar{N}^{1-\alpha}.
$$

Taking logs:

$$
\log Y_{t+1} = \mu + \log z_{t+1} + \alpha \log Y_t \\
= \mu + \rho \log z_t + \alpha \log Y_t + \varepsilon_{t+1}.
$$

where $\mu = \alpha \log s + (1 - \alpha) \log \bar{N}$
Implications: Output Persistence

\[ \log Y_{t+1} = \mu + \rho \log z_t + \alpha \log Y_t + \varepsilon_{t+1}. \]

- Output and technology together follow a (vector) \( AR(1) \).
- Can simplify further, using:
  \[ \log z_t = \log Y_t - \mu - \alpha \log Y_{t-1} \]

So then:

\[ \log Y_{t+1} = (1 - \rho)\mu + (\rho + \alpha) \log Y_t - \alpha \rho \log Y_{t-1} + \varepsilon_{t+1}. \]

- Output follows an \( AR(2) \) process.
- Output is persistent because of the TFP shocks and because of capital accumulation.
Output and TFP Comovements

Ouput (black) and TFP (red), $\rho = 0.7$

Ouput (black) and TFP (red), $\rho = 0.99$
Simulations from a Quantitative Version

- We have seen the qualitative behavior of the model, showing that the real business cycle model is consistent with the data.
- Apart from the special case we studied, to fully solve the model we need to use numerical methods.
- Calibrate the model: choose parameters to match some key economic data. Example: set $\beta$ so that steady state real interest rate matches US data.
- Program up on computer and simulate: use random number generator to draw technology shocks, feed them through the model.
- Compute correlations and volatilities and compare to US data.
Calibrating an RBC Model

This problem will show how to choose some parameters of a RBC model to match the data, a process known as calibration. Suppose preferences are given by:

\[ \sum_{t=0}^{\infty} \beta^t (1 + n)^t [(1 - a) \log c_t + a \log(1 - N_t)] \]

here \( n > 0 \) is the population growth rate and \( c_t \) and \( N_t \) are per capita consumption and hours. Suppose labor-augmenting technology grows at rate \( g \) so \( A_t = (1 + g)^t \). Thus the aggregate resource constraint is:

\[ c_t + I_t = (1 + g)^{(1-\alpha)t} k_t^{\alpha} N_t^{1-\alpha}, \]

where \( I_t \) is per capita investment an \( k_t \) is the per capita capital stock. Finally the law of motion for the capital in per capita terms is:

\[ (1 + g)(1 + n)k_{t+1} = (1 - d)k_t + I_t \]

Working directly with the social planner’s problem, find the first order condition for hours worked and also find the Euler equation for the optimal consumption allocation.
We will use that for Cobb-Douglas production $F_K = \alpha Y / K$, $F_N = (1 - \alpha) Y / N$. The Lagrangian is

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t (1 + n)^t \left\{ (1 - a) \log c_t + a \log(1 - N_t) - \lambda_t \left[ c_t + (1 + g)(1 + n)k_{t+1} - (1 - d)k_t - (1 - g)^{(1-a)t} k_t^\alpha N_t^{1-\alpha} \right] \right\}$$

where we have already substituted for investment. Now the FOC are

$$\frac{1 - a}{c_t} = \lambda_t$$

$$\frac{a}{1 - N_t} = (1 - \alpha) \lambda_t \frac{y_t}{N_t}$$

$$(1 + g)(1 + n) \lambda_t = \beta \lambda_{t+1} (1 + n) \left[ 1 - d + \alpha \frac{y_{t+1}}{k_{t+1}} \right]$$

Now let us consolidate these three equations by eliminating the $\lambda$s,

$$\frac{a}{1 - N_t} = (1 - \alpha) \frac{1 - a}{c_t} \frac{y_t}{N_t} \quad (1)$$

$$\left(1 + g\right) \frac{c_{t+1}}{c_t} = \beta \left[ 1 - d + \alpha \frac{y_{t+1}}{k_{t+1}} \right] \quad (2)$$
2. This model has a balanced growth path (BGP) in which hours worked $N_t$ is constant and all other per capita variables grow at the constant rate $g$, i.e. $k_{t+1} = (1 + g)k_t$ and so on. Using the two relations derived in part (a) and the law of motion for capital, find three equations relating the hours $N$, the capital/output ratio $k/y$, the consumption/output ratio $c/y$, and the investment/capital ratio $I/k$ to each other and the parameters of the model.

First, we divide the law of motion for capital by $k_t$, and use $k_{t+1}/k_t = 1 + g$ to obtain

$$(1 + g)^2(1 + n) = 1 - d + \frac{I}{k} \tag{3}$$

Then using (1),

$$\frac{a}{1 - N} = (1 - \alpha) \frac{1 - a y}{N} \frac{c}{c} \tag{4}$$

Finally from (2),

$$(1 + g)^2 = \beta \left[ 1 - d + \alpha \frac{y}{k} \right] \tag{5}$$
3. Suppose $\alpha = 0.4$, $n = 0.012$ and $g = 0.0156$, which are estimated from US data.

1. Given a value of $I/k = 0.076$ in the data, find a value of $d$ consistent with this in the BGP. Using (3), we obtain $d = 0.0321$.

2. Given a value of $k/y = 3.32$ and your value of $d$ find a value of $\beta$ from the BGP relations. Now using (5) and the previously obtained value of $d$, we can calculate $\beta = 0.9478$.

3. Given a value of $N = 0.31$ and $y/c = 1.33$ find a value of $a$ from the BGP relations. Finally from (4), $a = 0.640$. 
Figure 10.03  Small shocks and large cycles
Labor

Williams

Economics 312
Impulse Responses

![Graph showing impulse responses](image-url)
Figure 10.01 Actual versus simulated volatilities of key macroeconomic variables
Figure 10.02  Actual versus simulated correlations of key macroeconomic variables with GNP
Assessment of the Basic Real Business Model

- It accounts for a substantial amount of the observed fluctuations. Accounts for the covariances among a number of variables. Has some problems accounting for hours worked.
- Are fluctuations in TFP really productivity fluctuations?
- Factor utilization rates vary over the business cycle. During recessions, firms reduce the number of shifts. Similarly, firms are reluctant to fire trained workers.
- Neither is well-measured. They show up in the Solow residual.
- There is no direct evidence of technology fluctuations.
- Is intertemporal labor supply really so elastic?
- All employment variation in the model is voluntary, driven by intertemporal substitution.
- Deliberate monetary policy changes appear to have real effects.
Great Depression is a unique event in US history.
Timing 1929-1933.
Major changes in the US Economic policy: New Deal.
Can we use the theory to think about it?
<table>
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<tr>
<th>Year</th>
<th>$u$</th>
<th>$Y$</th>
<th>$C$</th>
<th>$I$</th>
<th>$G$</th>
<th>$i$</th>
<th>$\pi$</th>
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<td>0.6</td>
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Output, Inputs and TFP During the Great Depression

Theory

\[ \frac{\dot{z}}{z} = \frac{\dot{Y}}{Y} - \alpha \frac{\dot{K}}{K} - (1 - \alpha) \frac{\dot{N}}{N} \]

Data (1929=100)

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<tr>
<th>Year</th>
<th>(Y)</th>
<th>(N)</th>
<th>(K)</th>
<th>(z)</th>
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<td>92.7</td>
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<td>101.4</td>
<td>83.4</td>
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<tr>
<td>1933</td>
<td>65.3</td>
<td>73.5</td>
<td>98.4</td>
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</table>

Why did TFP fall so much?
Figure 1: Real Output, Consumption and Private Hours
(Per Adult, Index 1929 =100)
Predicted and Actual Output in 1929–39

Detrended Levels, With Initial Capital Stock in the Model Equal to the Actual Capital Stock in 1929
Potential Reasons

- Changes in Capacity Utilization.
- Changes in Quality of Factor Inputs.
- Changes in Composition of Production.
- Labor Hoarding.
- Increasing Returns to Scale.
Other Reasons for Great Depression

- Based on Cole and Ohanian (1999)
- Monetary Shocks: Monetary contraction, change in reserve requirements too late
- Banking Shocks: Banks that failed too small
- Fiscal Shocks: Government spending did rise (moderately)
- Sticky Nominal Wages: Probably more important for recovery
Cole and Ohanian (2001). Data (1929=100); data are detrended

<table>
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<th>Y</th>
<th>z</th>
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<tr>
<td>1939</td>
<td>73.2</td>
<td>103.1</td>
</tr>
</tbody>
</table>

Fast Recovery of $z$, slow recovery of output. Why?
Predicted and Actual Recovery of Output in 1934–39

Detrended Levels, With Initial Capital Stock in the Model Equal to the Actual Capital Stock in 1934

Predicted

Actual

Index

110

100

90

80

70

60

1930  1932  1934  1936  1938