Money

Exercise 1

Suppose that a household has preferences given by:

$$\log(C^m) + \log(C^c) + \beta [\log(C^{m'}) + \log(C^{c'})]$$

where $C^m$ is consumption of cash goods, $C^c$ is consumption of credit goods. The household doesn’t value leisure and supplies one unit of labor inelastically. The household starts with $M^-$ nominal money holdings and no nominal bond holdings ($B^- = 0$), and faces the CIA constraints:

$$PC^m \leq M^- \text{ and } P'C^{m'} \leq M^d$$

The household also faces nominal budget constraints:

$$PC^m + PC^c + B^d + M^d = M^- + PY$$
$$P'C^{m'} + P'C^{c'} = M^d + (1 + R)B^d + P'Y'$$

where we write $Y$ as the household’s real income. Find the household’s money demand function. What is the velocity of money here?
Exercise 2 (Midterm 1 Question 1)

1. Consider a variation on the Solow model where the savings rate is variable instead of constant. In particular suppose that as usual output is produced competitively via a Cobb-Douglas production function: $Y = K^\alpha N^{1-\alpha}$, and the population grows at the constant rate $n > 0$. Suppose that there is no depreciation or productivity growth ($\delta = g = 0$), and that total savings is given by $S_t = s(r)Y_t$, where:

$$s(r) = s \phi$$

where we introduce the constants $s > 0$ and $\phi > 0$. Thus $s(r)$ is increasing in $r$, so that higher real interest rates induce higher savings rates.

(a) Solve for the steady state equilibrium per-worker quantities of capital, output, and consumption (as a function of $\alpha, \phi, s, n$) if $\phi < \frac{\alpha}{1-\alpha}$.

**Solution:** As usual, we introduce the notation $k = K/N$, $y = Y/N = k^\alpha$. Then

$$\dot{K} = s(r)Y$$

$$\dot{k} = \frac{\dot{K}}{N} = \frac{K \dot{N}}{N} = s(r)k^\alpha - nk = s \phi k^\alpha - nk$$

From firm’s optimization problem

$$r = MPK = \alpha k^{\alpha-1}$$

Thus the fundamental equation of the Solow Model is

$$\dot{k} = s \phi k^{\alpha+\phi(\alpha-1)} - nk$$
The steady state is at $\dot{k} = 0$, where

$$k^* = \left( \frac{\bar{s}\alpha}{n} \right) \frac{1}{(1+\phi)(1-\alpha)}$$

$$y^* = (k^*)^\alpha = \left( \frac{\bar{s}\alpha}{n} \right) \frac{1}{1+\phi} \frac{1}{(1-\alpha)}$$

$$c^* = (1-s(r))y^* = \left[ 1 - n \left( \frac{\bar{s}\alpha}{n} \right) \frac{1}{1+\phi} \right] \left( \frac{\bar{s}\alpha}{n} \right) \frac{1}{(1+\phi)(1-\alpha)}$$

(b) What are the effects of an increase in the interest elasticity of savings $\phi$ on the per-worker quantities of capital, output, and consumption? Consider both the short-run and long-run effects, and continue to assume $\phi < \frac{\alpha}{1-\alpha}$ even after the increase.

**Solution:** First consider the long-run effects of an increase in $\phi$ on $k^*$, $y^*$, and $c^*$, we just need to check the sign of $\frac{\partial k^*}{\partial \phi}$, $\frac{\partial y^*}{\partial \phi}$, $\frac{\partial c^*}{\partial \phi}$.

Define $u(\phi) = \frac{\bar{s}\alpha}{n}$, $v(\phi) = \frac{1}{(1+\phi)(1-\alpha)}$, then

$$k^* = u(\phi)v(\phi) = e^{v(\phi)\ln u(\phi)}$$

Thus

$$\frac{\partial k^*}{\partial \phi} = \left( u(\phi)v(\phi) \right)' = \left( e^{v(\phi)\ln u(\phi)} \right)' = e^{v(\phi)\ln u(\phi)} (v(\phi)\ln u(\phi))' = u(\phi)v(\phi) \left( v'(\phi)\ln u(\phi) + v(\phi)\frac{u'(\phi)}{u(\phi)} \right)$$

Since $u'(\phi) = \frac{\bar{s}\alpha \ln \alpha}{n}$, $v'(\phi) = \frac{-1}{(1-\alpha)(1+\phi)}$, we have

$$\frac{\partial k^*}{\partial \phi} = \left( \frac{\bar{s}\alpha}{n} \right) \frac{1}{(1+\phi)(1-\alpha)} \left( \frac{-1}{(1-\alpha)(1+\phi)^2} \ln \frac{\bar{s}\alpha}{n} + \frac{1}{(1+\phi)(1-\alpha)\ln \alpha} \right)$$

If $\ln \alpha > \frac{1}{(1+\phi)} \ln \frac{\bar{s}\alpha}{n}$, $\frac{\partial k^*}{\partial \phi} > 0$, $k^*_{NEW} > k^*$, $y^*_{NEW} > y^*$;

if $\ln \alpha < \frac{1}{(1+\phi)} \ln \frac{\bar{s}\alpha}{n}$, $\frac{\partial k^*}{\partial \phi} < 0$, $k^*_{NEW} < k^*$, $y^*_{NEW} < y^*$. 
Similarly,

\[
\frac{\partial c^*}{\partial \phi} = \alpha \left( \frac{\pi \alpha^\phi}{n} \right) \frac{1}{(1+\phi)(1-\alpha)} \left( \frac{-1}{(1-\alpha)(1+\phi)^2} \ln \frac{\pi \alpha^\phi}{n} + \frac{1}{(1+\phi)(1-\alpha) \ln \alpha} \right)
\]

\[
- n \left( \frac{\pi \alpha^\phi}{n} \right) \frac{1}{(1+\phi)(1-\alpha)} \left( \frac{-1}{(1-\alpha)(1+\phi)^2} \ln \frac{\pi \alpha^\phi}{n} + \frac{1}{(1+\phi)(1-\alpha) \ln \alpha} \right)
\]

\[
= \frac{1}{(1+\phi)(1-\alpha)} \left( \frac{-1}{(1+\phi) \ln \frac{\pi \alpha^\phi}{n} + \ln \alpha} \right) \left( \frac{\pi \alpha^\phi}{n} \right) \frac{1}{(1+\phi)(1-\alpha)} \left( \alpha - n \left( \frac{\pi \alpha^\phi}{n} \right) \right)
\]

If \( \left( \frac{-1}{(1+\phi) \ln \frac{\pi \alpha^\phi}{n} + \ln \alpha} \right) \left( \alpha - n \left( \frac{\pi \alpha^\phi}{n} \right) \right) > 0 \), \( \frac{\partial c^*}{\partial \phi} > 0 \), \( c^*_{\text{NEW}} > c^* \);

if \( \left( \frac{-1}{(1+\phi) \ln \frac{\pi \alpha^\phi}{n} + \ln \alpha} \right) \left( \alpha - n \left( \frac{\pi \alpha^\phi}{n} \right) \right) < 0 \), \( \frac{\partial c^*}{\partial \phi} < 0 \), \( c^*_{\text{NEW}} < c^* \).

As for the short run, capital, output and consumption will converge to the new steady state.

We can also draw the graph as follows. Suppose \( 0 < \alpha < 1 \), and \( g(\phi) = s(r) y = \bar{s} \alpha^\phi k^{\alpha+\phi(\alpha-1)} \) is the actual investment, \( nk \) is the break-even investment, then \( g'(\phi) = \bar{s} \alpha^\phi k^{\alpha+\phi(\alpha-1) \ln \alpha k^{\alpha-1}} \).

When \( \alpha k^{\alpha-1} > 1 \iff k < \alpha^{-1} \), \( g'(\phi) > 0 \), the new actual investment lies above the old one;

When \( \alpha k^{\alpha-1} < 1 \iff k > \alpha^{-1} \), \( g'(\phi) < 0 \), the new actual investment lies below the old one.
(c) Now suppose $\phi = \frac{\alpha}{1-\alpha}$. How does this change the model? What are the effects of an increase in the saving fraction $\bar{s}$ on capital and output now?

Solution: If $\phi = \frac{\alpha}{1-\alpha}$, then

$$\dot{k} = \bar{s}\alpha^{\frac{\alpha}{1-\alpha}} - nk$$

The steady state is at $\dot{k} = 0$, where

$$k^* = \frac{\bar{s}\alpha^{\frac{\alpha}{1-\alpha}}}{n}$$

$$y^* = (k^*)^\alpha = \left(\frac{\bar{s}\alpha^{\frac{\alpha}{1-\alpha}}}{n}\right)^\alpha$$

An increase in $\bar{s}$ increases $k^*$ and $y^*$ in the long run, and in the short run, capital and output will converge to the new steady state.