Dynamic Model

Household Preferences

\[ V_0 = \sum_{t=0}^{\infty} \beta^t u(c_t) \]

\( \beta \in (0, 1) \) discount factor measures household’s degree of impatience. Define \( \beta = \frac{1}{1+\theta} \), where \( \theta \) is discount rate

Technology

- production is:

\[ y_t = F(k_t, 1) \equiv F(k_t) \]

- Diminishing marginal returns in \( k \):

\[ F'(k) > 0, F''(k) < 0 \]

- Inada conditions:

\[ \lim_{k \to 0} F'(k) = +\infty, \lim_{k \to \infty} F'(k) = 0 \]
Investment

• Capital depreciates at rate $\delta$, and investment at $t$ increases $k_{t+1}$:

• We abstract from government spending, so the feasibility or goods market clearing condition now includes investment and consumption:

• Combining equations gives us the tradeoff between consumption and capital:

Optimal Allocation

• Solving social planner’s problem:

$$\max_{(c_t,k_{t+1})} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

$$s.t. \quad c_t = F(k_t) - k_{t+1} + (1-\delta)k_t, \forall t, \quad k_0 \text{ given}$$

• Form the Lagrangian with multipliers $\{\lambda_t\}$ on the constraints:

• First order conditions for any $c_t$, and for $k_{t+1}$, $t > 0$:

• Combine the two optimality conditions to get the Euler equation:

Optimal Steady State
Phase Diagram

We will analyze the joint dynamics of $c_t, k_t$. In any period $t$, $k_t$ is given and $c_t$ is chosen optimally (as is $k_{t+1}$).

1. Dynamics of consumption
   
   • In steady state, $c_{t+1} = c_{t+1} - c_t = 0$, and
     
     $F'(k^*) = \delta + \theta$
   
   • If $k < k^*$, then $F'(k) > F'(k^*)$, so to satisfy Euler equation we need $u'(c_{t+1}) < u'(c_t)$ and so $c_{t+1} > c_t$. Similarly if $k > k^*$, $\Delta c < 0$.

   ![Dynamics of consumption](image)

2. Dynamics of capital
   
   • A key equation of the model is
     
     $c_t = F(k_t) - k_{t+1} + (1-\delta)k_t$
   
     If $c_t < F(k_t) - \delta k_t$, then $\Delta k_{t+1} < 0$, similarly, if $c_t > F(k_t) - \delta k_t$, then $\Delta k_{t+1} > 0$.
   
   • In steady state, $k_{t+1} = k_{t+1} - k_t = 0$, and
     
     $c = F(k) - \delta k$
3. Phase Diagram
Exercise 1

Consider the optimal growth model from class, but add government spending. That is, there is now a specified amount of government spending $G$ which must be funded every period via lump sum taxes.

(a) Compared to the case of no government spending, how does having $G > 0$ affect the optimal steady state levels of consumption and capital $c^*$ and $k^*$?

(b) How are the dynamics affected? That is, suppose the economy is initially in the steady state $(k_0^*, c_0^*)$ associated with $G = 0$. Then there is announcement that there will be government spending $G > 0$ for all future dates. How do consumption and capital respond, both immediately upon the announcement and then in the succeeding periods? What about the real interest rate?

(c) Suppose there is announcement that there will be government spending $G > 0$ that is temporary. For simplicity, assume that the terminal date is known with certainty. How do consumption and capital respond, both immediately upon the announcement and then in the succeeding periods? What about the real interest rate?