Problem 1 (Production Functions)

(a) The isoquants for each of the three production functions are shown below:

- \( f(K, L) = K^2 L \)

- \( f(K, L) = K^{\frac{1}{3}} L^{\frac{1}{3}} \)

- \( f(K, L) = 2K + L \)
(b) The marginal productivity of capital, $MP_K$, tells us by how many units output would increase if capital input were increase by one unit (machine). (Mathematically, $MP_K$ is the partial derivative of the production function; the larger the change in capital the further the approximation gets from actual changes in output.)

The marginal productivity of labor, $MP_L$, tells us how much additional output we get from increasing labor input by one unit (worker).

(c) Marginal productivity of capital with $L = 1$:

- $MP_K$ when $f(K, L) = K^2L$:

$$MP_K = \frac{\partial f(K, L)}{\partial K} = 2KL \quad (MP_K \text{ is increasing in } K)$$

With $L = 1$, $MP_K = 2K$:

- $MP_K$ when $f(K, L) = K^{\frac{1}{2}}L^{\frac{1}{2}}$:

$$MP_K = \frac{\partial f(K, L)}{\partial K} = \frac{1}{3}K^{-\frac{3}{2}}L^{\frac{3}{2}} \quad (MP_K \text{ is decreasing in } K)$$

With $L = 1$, $MP_K = \frac{1}{3}K^{-\frac{3}{2}}$.
• $MP_K$ when $f(K, L) = 2K + L$:

$$MP_K = \frac{\partial f(K, L)}{\partial K} = 2 \quad (MP_K \text{ is constant in } K)$$

With $\overline{L} = 1$, $MP_K = 2$:

(d) Marginal productivity of labor with $\overline{K} = 2$:

• $MP_L$ when $f(K, L) = K^2L$:

$$MP_L = \frac{\partial f(K, L)}{\partial L} = K^2 \quad (MP_L \text{ is constant in } L)$$

With $\overline{K} = 2$, $MP_L = 4$: 
• \( MP_L \) when \( f(K, L) = K^{\frac{1}{3}} L^{\frac{1}{3}} \):

\[
MP_L = \frac{\partial f(K, L)}{\partial L} = \frac{1}{3} K^{\frac{1}{3}} L^{-\frac{2}{3}} \quad (MP_L \text{ is decreasing in } L)
\]

With \( K = 2 \), \( MP_L = \frac{1}{3} 2^{\frac{1}{3}} L^{-\frac{2}{3}} \):

\[
MP_L(2, L) = \frac{1}{3} 2^{\frac{1}{3}} L^{-\frac{2}{3}}
\]

• \( MP_L \) when \( f(K, L) = 2K + L \):

\[
MP_L = \frac{\partial f(K, L)}{\partial L} = 1 \quad (MP_L \text{ is constant in } L)
\]

With \( K = 2 \), \( MP_L = 1 \):

\[
MP_L(2, L) = 1
\]

(e) Returns to scale:

• Constant Returns to Scale (CRS), \( f(\lambda K, \lambda L) = \lambda f(K, L) \): The means that doubling all inputs leads to a doubling of output (or tripling inputs triples outputs, etc.) An example might be pastry making at a bakery, where twice as much of all inputs (\( L \): pastry chefs, \( K \): countertops, ovens, and butter, flour, eggs, etc.) leads to twice as much output (the pastries). Also, the Varian textbook mentions data centers: A thousand times as many data centers (inputs) leads to a thousand times as many webpages served (output).
• Decreasing Returns to Scale (DRS), \( f(\lambda K, \lambda L) < \lambda f(K, L) \): A doubling of inputs results in \emph{less than} double the output. As the Varian text notes, DRS is usually a short-run phenomenon where in fact there is some other input that \emph{is} held fixed (otherwise a firm could at least replicate a process and achieve CRS). For instance, in farming, a doubling of capital equipment and labor does not lead to a doubling of output so in that case we’d say there is DRS, but this is really because one of the inputs—land—might actually be fixed.

• Increasing Returns to Scale (IRS), \( f(\lambda K, \lambda L) > \lambda f(K, L) \): A doubling of inputs results in \emph{more than} a doubling of output. Again, here Varian gives a nice example: An oil pipeline. “If we double the diameter of a pipe, we use twice as much materials, but the cross section of the pipe goes up by a factor of 4. Thus we will likely be able to pump more than twice as much oil through it [up to a certain point].”

(f) Let’s see whether our three production functions exhibit CRS, DRS, or IRS for \( \lambda > 1 \):

• \( f(K, L) = K^2L \):
  \[
  f(\lambda K, \lambda L) = (\lambda K)^2\lambda L = \lambda^3 f(K, L) > \lambda f(K, L) \implies IRS
  \]

• \( f(K, L) = K^{\frac{1}{3}}L^{\frac{1}{3}} \):
  \[
  f(\lambda K, \lambda L) = (\lambda K)^{\frac{1}{3}}(\lambda L)^{\frac{1}{3}} = \lambda^{\frac{1}{3}+\frac{1}{3}}K^{\frac{1}{3}}L^{\frac{1}{3}} = \lambda^{\frac{2}{3}} f(K, L) < \lambda f(K, L) \implies DRS
  \]

• \( f(K, L) = 2K + L \):
  \[
  f(\lambda K, \lambda L) = 2(\lambda K) + \lambda L = \lambda(2K + L) = \lambda f(K, L) \implies CRS
  \]
Problem 2 (Profit Maximization in the Short Run)

(a) The profit of GMC is total revenue \((p \cdot f(K, L))\) minus cost \((w_L L)\):

\[
\pi = p \cdot f(K, L) - w_L \cdot L \quad \text{and} \quad K = 16 \quad \Rightarrow \quad \pi = 8pL^{\frac{1}{2}} - w_L L.
\]

Since capital is fixed, we are in the short run and costs include only the variable costs \(w_L L\).

(b) Total revenue, \(p \cdot f(K, L)\), and labor cost, \(w_L L\), are shown below for \(p = 1\) and \(w_L = 2\):

\[
\begin{align*}
\text{TC} &= w_L L \\
\text{TR} &= p \cdot f(K, L) \\
\pi &= \text{TR} - \text{TC}
\end{align*}
\]

(c) A well-behaved function \(\pi(x)\) is flat at the point at which it attains a local maximum (increasing to the left, flat, then decreasing to the right). Since the derivative is zero when \(\pi(x)\) is flat, finding the \(x\) at which \(\pi'(x) = 0\) tells us where a local maximum is. This is what we call the first-order condition. (We can assume for the profit functions we’ll be working with that there is only one local maximum and that it is the global maximum.) Warning: A function is also flat where it attains a minimum, therefore we should check whether actually our \(x\) is not minimizing the value of the function (this is the second-order condition: \(\pi''(x) > 0\) means it’s a minimum, \(\pi''(x) < 0\) means it’s a maximum). This won’t be an issue in our application to maximization of profit function though.

(d) Setting the derivative of the profit function to zero we have

\[
\frac{\partial \pi}{\partial L} = 0 \quad \Rightarrow \quad p \cdot \frac{\partial f(K, L)}{\partial L} - w_L = 0 \quad \Rightarrow \quad MP_L = \frac{w_L}{p}.
\]

Alternatively, we can see this using the production function \(f(K, L) = 8L^{\frac{1}{2}}\) for \(K = 16\). We then have \(\pi = 8pL^{\frac{1}{2}} - w_L L\), so

\[
\frac{\partial \pi}{\partial L} = \frac{1}{2} 8pL^{-\frac{1}{2}} - w_L.
\]
and setting this equal to zero (our first order condition), we get
\[
\frac{\partial \pi}{\partial L} = 0 \implies \frac{1}{2} 8pL^{-\frac{1}{2}} - w_L = 0 \implies 4L^{-\frac{1}{2}} = \frac{w_L}{p}.
\]
(1)

Since the marginal product of labor is \(MP_L = \frac{\partial f(K,L)}{\partial L} = 4L^{-\frac{1}{2}}\), in equation (1) we in fact found the condition that \(MP_L = \frac{w_L}{p}\).

The intuition is that a firm should hire as long as marginal benefits (additions to output \(MP_L\)) are greater than the marginal costs of doing so (real wage \(\frac{w_L}{p}\)), up to the point where additional benefits and costs are exactly equal (\(MP_L = \frac{w_L}{p}\)). Past this point, a firm shouldn’t hire any more labor since \(MP_L < \frac{w_L}{p}\) (since \(MP_L\) is always decreasing).

(e) To find the optimal level of labor, we can use the condition we found in part (d) in equation (1): \(MP_L = \frac{w_L}{p}\) or \(4L^{-\frac{1}{2}} = \frac{w_L}{p}\). Solving for \(L\) we get the labor demand curve:

\[
L^D = \left( \frac{4p}{w_L} \right)^2
\]

- For \(p = 1, w_L = 8\), we have \(L^* = \left( \frac{4}{8} \right)^2 = \frac{1}{4}\)
- For \(p = 1, w_L = 4\), we have \(L^* = \left( \frac{4}{4} \right)^2 = 1\)
- For \(p = 1, w_L = 2\), we have \(L^* = \left( \frac{4}{2} \right)^2 = 4\)

These points are shown in the graph below:
We know from part (a) what profit is associated with any $p$, $w_L$, and $L^*$: $\pi = 8pL^{\frac{1}{2}} - w_LL$, so we have:

- For $p = 1$, $w_L = 8$, $L^* = \frac{1}{4}$, we have $\pi = 8(1)(\frac{1}{4})^{\frac{1}{2}} - (8)(\frac{1}{4}) = 2$
- For $p = 1$, $w_L = 4$, $L^* = 1$, we have $\pi = 8(1)(1)^{\frac{1}{2}} - (4)(1) = 4$
- For $p = 1$, $w_L = 2$, $L^* = 4$, we have $\pi = 8(1)(4)^{\frac{1}{2}} - (2)(4) = 8$

**Problem 3 (Labor Market)**

(a) Kate’s (perfectly inelastic) labor supply, $L^S = 12$ is shown below:

(b) We had that labor demand was given by $L^D = \left(\frac{4p}{w_L}\right)^2$. We get the equilibrium wage rate by equating $L^S = L^D$ and solving for $\frac{w_L}{p}$.

$$L^S = L^D \implies 12 = \left(\frac{4p}{w_L}\right)^2 \implies \frac{w_L}{p} = \left(\frac{4}{3}\right)^{\frac{1}{2}} \approx 1.15$$
(c) At a hypothetical wage above what we found in part (b), the hours of labor demanded is less than the supply at that wage of \( L^S = 12 \). This excess supply is unemployment. Since there is willingness to work at lower wages, the wage offered would fall, bringing the excess supply (unemployment) to zero. (Same is true at a point below the equilibrium wage we found: There would be excess demand, so to attract more workers the wages would be bid up to the point where there is no excess demand.)

(d) Now with \( L^S = 12 \), equating \( L^S = L^D \) and solving for \( \frac{w_L}{p} \) we get:

\[
L^S = L^D \quad \Rightarrow \quad 8 = \left( \frac{4p}{w_L} \right)^2 \quad \Rightarrow \quad \frac{w_L}{p} = 2^{\frac{1}{2}} \approx 1.41
\]

(e) At this price floor of \( \frac{w_L}{p} = 2 \), we have that \( L^S = 8 \) (unchanged) but now \( L^D = \left( \frac{4p}{w_L} \right)^2 = (4 \frac{1}{2})^2 = 4 \). The unemployment rate is now \( \frac{L^S - L^D}{L^S} = \frac{8 - 4}{8} = .5 \) or 50% unemployment. (The unemployment rate was previously zero: \( \frac{L^S - L^D}{L^S} = 0 \) since in the market we have \( L^S = L^D = 8 \).
Problem 4 (The Long Run)

(a) To determine the returns to scale, we must compare \( f(\lambda K, \lambda L) \) to \( \lambda f(K, L) \) for any number \( \lambda > 1 \):

\[
f(\lambda K, \lambda L) = (\lambda K)^{\frac{1}{3}}(\lambda L)^{\frac{1}{3}} = \lambda^{\frac{2}{3}} f(K, L) < \lambda f(K, L) \implies DRS
\]

So since \( f(\lambda K, \lambda L) < \lambda f(K, L) \), this function exhibits decreasing returns to scale.

(b) The profit function in terms of \( K \) and \( L \) is given by:

\[
\pi = p \cdot f(K, L) - (w_L \cdot L + w_K \cdot K).
\]

With \( p = 1 \), \( w_K = 2 \), and \( w_L = 1 \),

\[
\pi = f(K, L) - (L + 2K).
\]

(c) First, we’ll find the optimal combination of inputs \( K \) and \( L \). From our profit function above, setting the partial derivatives with respect to \( K \) and \( L \), we get secrets of happiness

\[
MP_K = \frac{w_K}{p} \quad \text{and} \quad MP_L = \frac{w_L}{p}
\]

and substituting in the marginal productivities of capital and labor as well as prices, this is equivalent to

\[
\frac{1}{3}K^{-\frac{2}{3}}L^{\frac{1}{3}} = 2 \quad \text{and} \quad \frac{1}{3}K^{\frac{1}{3}}L^{-\frac{2}{3}} = 1.
\]  \hspace{1cm} (2)

Dividing the first equation by the second, we get

\[
\frac{\frac{1}{3}K^{-\frac{2}{3}}L^{\frac{1}{3}}}{\frac{1}{3}K^{\frac{1}{3}}L^{-\frac{2}{3}}} = \frac{2}{1} \implies \frac{L}{K} = 2 \implies L = 2K.
\]
so we will be using $K$ and $L$ such that $L = 2K$.

Plugging $L = 2K$ into the first equation in (2), we have

$$\frac{1}{3} K^{-\frac{2}{3}} (2K)^{\frac{1}{3}} = 2 \implies K = \left(3 \times 2^\frac{2}{3}\right)^{-3} = \frac{1}{108}$$

and so

$$L = 2K \implies L = 2 \times \frac{1}{108} = \frac{1}{54}.$$ 

Given these two values, the optimal level of output is

$$y = f(K, L) = \left(\frac{1}{108}\right)^{\frac{1}{3}} \left(\frac{1}{54}\right)^{\frac{1}{3}} = \frac{1}{18}.$$ 

and the profit associated with this level of output and the prices given is

$$\pi = p \cdot f(K, L) - (w_L \cdot L + w_K \cdot K) = 1 \cdot \frac{1}{18} - (2 \cdot \frac{1}{108} + 1 \cdot \frac{1}{54}) = \frac{1}{54}.$$ 

(d) The condition for cost minimization is

$$TRS = -\frac{w_K}{w_L}$$

Since for the technical rate of substitution $TRS$ we have

$$TRS = -\frac{MP_K}{MP_L} = -\frac{\frac{1}{3} K^{-\frac{2}{3}} L^{\frac{1}{3}}}{\frac{1}{3} K^{\frac{1}{3}} L^{-\frac{2}{3}}} = -\frac{L}{K} = -\frac{1/54}{1/108} = -2$$

and

$$-\frac{w_K}{w_L} = -\frac{2}{1} = -2$$

we indeed are satisfied the cost minimization condition $TRS = -\frac{w_K}{w_L}$. 