Problem 1 (20p). (Intertemporal choice)
Frank works as a consultant. His income when young is $4000 (period 1) and $16000 when old (period 2), the interest rate is $r = 100\%$.

a) In the graph depict Frank’s budget set. Mark all the bundles on the budget line that involve saving and the ones that involve borrowing. Find analytically PV and FV of income and show it in the graph

$$PV = 4000 + \frac{16000}{1 + 1} = 4000 + 8000 = 12000$$

$$FV = (1 + 1)4000 + 16000 = 24000$$

b) Frank’s intertemporal preferences are given by

$$U(C_1; C_2) = \ln C_1 + \frac{1}{1 + \delta} \ln C_2$$

where the discount factor is $\delta = 100\%$. Using the magic formula, find the optimal consumption plan $(C_1, C_2)$ and how much Frank borrows or saves (three numbers).

$$C_1 = \frac{1}{1 + \frac{1}{2}} \frac{24000}{2} = \frac{2}{3} 12000 = 8000$$

$$C_2 = \frac{1}{1 + \frac{1}{2}} \frac{24000}{1} = \frac{1}{3} 24000 = 8000$$

$$S = 4000 - 8000 = -4000$$

Frank borrows $-4000$

c) Is Frank smoothing his consumption? (yes or no answer + one sentence).
Yes, because $C_1 = C_2$. This is because $\delta = r$.

Problem 2 (30p). (Edgeworth box, and equilibrium)
Consider an economy with apples and oranges. Peter is initially endowed with five apples and ten oranges $\omega^p = (5, 10)$. Amanda’s endowment is $\omega^A = (10, 5)$.

a) Plot the Edgeworth box and mark the allocation representing the initial endowment.

b) Describe the concept of Pareto efficiency (one intuitive sentence). Peter and Amanda have the same utility function

$$U^i(C_1, C_2) = 2 \ln (C_1) + 2 \ln (C_2).$$
Verify whether the endowment allocation is (or is not) Pareto efficient (use values of $MRS$ in your argument). Illustrate your argument geometrically in the Edgeworth Box from a).

The endowment allocation is not Pareto efficient, as at this allocation the slopes of indifference curves are not tangent to each other

$$MRS^P = \frac{C_2^P}{C_1^P} = \frac{10}{5} = 2$$

$$MRS^A = \frac{C_2^A}{C_1^A} = \frac{5}{10} = \frac{1}{2}$$

and hence they do not coincide (see graph above).

c) Find analytically the competitive equilibrium (six numbers) and show it in the Edgeworth box. Find some other prices that define competitive equilibrium (two numbers).

We normalize $p_2 = 1$

$$C_1^P = \frac{1}{2} \frac{5p_1 + 10}{p_1}$$

$$C_1^A = \frac{1}{2} \frac{10p_1 + 5}{p_1}$$

and market clearing condition gives

$$\frac{1}{2} \frac{5p_1 + 10}{p_1} + \frac{1}{2} \frac{10p_1 + 5}{p_1} = 15$$

From which one can find price $p_1 = 1 = p_2$. Equilibrium consumption is $C_1^A = C_1^B = 7.5$ and $C_2^A = C_2^B = 7.5$.

Other prices: $p_1 = p_2 = 2$

d) Argue that competitive markets allocate resources efficiently (give two numbers and compare them).

Allocation in competitive equilibrium is Pareto efficient as $MRS$ of both agents are the same

$$MRS^P = \frac{C_2^P}{C_1^P} = \frac{7.5}{7.5} = 1$$

$$MRS^A = \frac{C_2^A}{C_1^A} = \frac{7.5}{7.5} = 1$$

Problem 3 (20p). (Short questions)

a) The Bernoulli utility function is given by $u(c) = c^2$ and two states of the world are equally likely. Find the corresponding von Neuman Morgenstern (expected) utility function (give formula). Is such agent risk neutral, risk loving or risk averse? (one sentence). Find the expected value and the certainty equivalent of a lottery $(2, \sqrt{28})$ (two numbers). Which is bigger and why (one sentence) (Hint: when calculating expected value of the lottery, use that $\sqrt{28} \approx 5.3$).

Expected Utility function is given by

$$U(c_1, c_2) = \frac{1}{2} c_1^2 + \frac{1}{2} c_2^2$$
Agent is risk loving as Bernoulli utility function is convex.

\[ E(L) = \frac{1}{2} \cdot 2 + \frac{1}{2} \cdot 3.3 = 3.6 \]

Certainty equivalent can be found as

\[ (CE)^2 = U(2, \sqrt{28}) = 2 + 14 = 16 \]

and hence

\[ CE = 4 > E(L) \]

This is because risk loving agent derives extra utility from uncertainty regarding the outcome.

b) Derive the formula for perpetuity

\[
PV = \frac{x}{1+r} + \frac{x}{(1+r)^2} + \frac{x}{(1+r)^3} + \ldots = \frac{x}{1+r} + \frac{1}{1+r} \left( \frac{x}{1+r} + \frac{x}{(1+r)^2} + \ldots \right) = \frac{x}{1+r} + \frac{1}{1+r} PV
\]

Solving for PV gives

\[ PV = \frac{x}{r} \]

c) You will live for 4 periods. You would like to maintain the constant level consumption throughout your life C. How much can you consume if in the first three periods you earn $1500? The interest rate is r=100%? (one number)

\[
\frac{C}{r} \left( 1 - \left( \frac{1}{1+r} \right)^4 \right) = 3000 \left( 1 - \left( \frac{1}{1+r} \right)^3 \right)
\]

\[
\frac{15}{16} C = 3000 \cdot \frac{7}{8}
\]

\[ C = 2800 \]

Problem 4 (30p). (Producers)

Consider a producer that has the following technology

\[ y = 8K^{\frac{1}{4}}L^{\frac{1}{2}} \]

a) What returns to scale are represented by this production function? (choose between CRS, IRS or DRS; you do not have to prove it).

DRS

b) (Short run) Assume that \( K = 1 \) and the firm cannot change it in a short run. Derive a condition for optimal demand for labor. Explain intuitively its economic meaning. (one sentence). \( MPL = \frac{w_k}{p} \)

\[
\frac{w_k}{p} = 4L^{-\frac{1}{2}}
\]

Last worker produces as much as he gets in terms of wage.

c) Suppose that labor supply is inelastic and given by \( L^* = 16h \). Find analytically and on the graph the equilibrium wage rate.

\[
\frac{w_k}{p} = 4(16)^{-\frac{1}{2}} = 1
\]
d) Find the unemployment rate with the minimal (real) wage given by $w_L/p = 4/3$. 
With minimal wage rate the demand for labor is 

$$4/3 = 4L^{-\frac{1}{2}} \Rightarrow L = 9$$

and hence unemployment rate is 

$$UR = \frac{16 - 9}{16} = \frac{7}{16}$$

e) Suppose $w_L = 1, w_K = 2$. Derive the cost function $C(y)$, assuming that you can adjust both $K$ and $L$, and plot it on the graph. Relate the shape of your cost function to the returns to scale. (Hint: the constants in this last questions are not round numbers)

$$8K^\frac{4}{y}L^{\frac{1}{2}}$$

$$TRS = -\frac{1}{2} \frac{L}{K} = -\frac{2}{1}$$

and hence

$$L = 4K$$

It follows that

$$K = \left(\frac{1}{16^y}\right)^{\frac{4}{7}}$$

and

$$L = 4 \left(\frac{1}{16^y}\right)^{\frac{4}{7}}$$

It follows that

$$c(y) = 4 \left(\frac{1}{16^y}\right)^{\frac{4}{7}} + 2 \left(\frac{1}{16^y}\right)^{\frac{4}{7}} = 6 \left(\frac{1}{16^y}\right)^{\frac{4}{7}}$$

The function is convex as we have DRS.

**Bonus Problem. (extra 10 points)**

Give examples of production functions with perfect complements and perfect substitutes that are characterized by increasing and decreasing returns to scale.

Perfect complements

$$y = \min (2K, 7L)^2$$ (IRS)

$$y = \min (2K, 7L)^{\frac{2}{7}}$$ (DRS)

Perfect substitutes

$$y = (2K + 7L)^2$$ (IRS)

$$y = [2K + 7L]^{\frac{2}{7}}$$ (DRS)