Problem 1 (25p). (Labor supply)

a) The slope of the budget set is a real wage rate \( w/p \) that tells how many steaks Peter can get for every hour he works.

\[ \text{slope} = \frac{20}{4} = 5 \]

b) This is a Cobb-Douglass utility function therefore we can find his optimal choice \( R, C \) using our "magic" formula, we have derived earlier in our class. The values of parameters are:

\[ a = 3, b = 1, \]

and hence the relaxation time and consumption is

\[ R = \frac{a m}{a + b p_1} = \frac{3 \times 24w}{4 w} = 18 \]
\[ C = \frac{b m}{a + b p_1} = \frac{1 \times 24w}{4 p} = \frac{6w}{p} \]

For \( w = 100 \), and \( p = 5 \) we have \( R = 18 \) and \( C = 120 \)

In such case the labor supply is given by

\[ LS = 24 - R = 6 \]

c) The labor supply function is inelastic with respect to \( \frac{w}{p} \). The reason for that is that the substitution effect (higher real wage makes leisure more expensive relative to consumption encouraging work) is offset by income effect (the higher income makes leisure more attractive)

Problem 2 (35p).

a) 

b) The endowment allocation is not Pareto efficient, as at this allocation the slopes of indifference curves

\[ MRS^{P} = \frac{C_2^P}{C_1^P} = \frac{300}{100} = 3 \]
\[ MRS^{A} = \frac{C_2^A}{C_1^A} = \frac{100}{100} = 1 \]

and hence they do not coincide (see graph above).
c) We normalize $p_2 = 1$. The optimal consumption today is
\[
C^P_1 = \frac{1}{2} \frac{100p_1 + 300}{p_1}, \quad C^A_1 = \frac{1}{2} \frac{100p_1 + 100}{p_1}
\]
Market clearing condition implies that
\[
\frac{1}{2} \frac{100p_1 + 300}{p_1} + \frac{1}{2} \frac{100p_1 + 100}{p_1} = 200
\]
or
\[
p_1 = 2
\]
and hence
\[r = 100\%
\]
At this price consumption is given by
\[
C^P_1 = \frac{1}{2} \frac{2 \times 100 + 300}{2} = 125 \quad \text{and} \quad C^A_1 = 200 - 125 = 75
\]
and
\[
C^P_2 = \frac{1}{2} \frac{2 \times 100 + 300}{1} = 250 \quad \text{and} \quad C^A_2 = 400 - 250 = 150
\]
Hence allocation $C^P = (125, 250)$, $C^A = (75, 125)$ and interest rate $r = 100\%$ is an equilibrium.
d) Savings are given by
\[
s^P = \omega^P_1 - C^P_1 = 100 - 125 = -25
\]
hence Peter is borrowing $25$.
\[
s^A = \omega^A_1 - C^A_1 = 100 - 75 = 25
\]
and Amanda is saving $25$.
e) \[
MRS^P = \frac{C^P_2}{C^P_1} = \frac{250}{125} = 2
\]
\[
MRS^A = \frac{C^A_2}{C^A_1} = \frac{150}{75} = 2
\]
The equilibrium allocation is Pareto efficient as the indifference curves are tangent (they have the same slope $MRS$).
f) \[
PV = 100 + 100 \cdot \frac{1}{1 + 100\%} = 100 + \frac{100}{2} = 150
\]
\[
FV = 100 \times (1 + 100\%) + 100 = 200 + 100 = 300
\]

Problem 3 (15p). (Short questions)
a) Using annuity formula
\[
2800 = \frac{x}{1} \left( 1 - \left( \frac{1}{2} \right)^3 \right) = \frac{7}{8} \Rightarrow x = \frac{8}{7} \times 2800 = 8 \times 400 = 3200
\]
b) Expected value of the lottery is
\[
E(L) = \frac{1}{2} \times 36 + \frac{1}{2} \times 25 = 18 + 12 \frac{1}{2} = 30 \frac{1}{2}
\]
The von Neumann Morgenstern lottery is

\[ U = \frac{1}{2} \sqrt{36} + \frac{1}{2} \sqrt{25} = \frac{1}{2} \times 6 + \frac{1}{2} \times 5 = \frac{11}{2} \]

the Certainty equivalent is

\[ \sqrt{CE} = \frac{11}{2} \Rightarrow CE = \left( \frac{11}{2} \right)^2 = 30 \frac{1}{4} \]

\( CE < E(L) \) because the agent is risk averse, and hence is willing to accept lower payment for sure.

c) You are willing to pay

\[ PV = \frac{500}{0.1} = 5000 \]

Problem 4 (25p). (Producers)
a) Suppose \( \lambda > 1 \). Then

\[ F(\lambda K, \lambda L) = (\lambda K)^{\frac{1}{\lambda}} (\lambda L)^{\frac{1}{\lambda}} = \lambda^{\frac{1}{\lambda}} K^{\frac{1}{\lambda}} L^{\frac{1}{\lambda}} < \lambda K^{\frac{1}{\lambda}} L^{\frac{1}{\lambda}} = \lambda F(K, L) \]

hence we have DRS.

b) We use two conditions

\[ MPK = \frac{w_K}{p} \]
\[ MPL = \frac{w_L}{p} \]

which become \( \frac{1}{6} \)

\[ \frac{1}{6} K^{-\frac{5}{6}} L^\frac{5}{6} = \frac{1}{6} \]
\[ \frac{1}{6} K^\frac{1}{6} L^{-\frac{5}{6}} = \frac{1}{6} \]

Implying

\[ \frac{K}{L} = 1 \Rightarrow K = L \]

Plugging back in the two secrets of happiness

\[ K^{-\frac{5}{6}} K^{\frac{1}{6}} = K^{-\frac{2}{3}} = 1 \Rightarrow K = 1 \]
\[ L^{\frac{5}{6}} L^{-\frac{5}{6}} = L^{\frac{1}{3}} = 1 \Rightarrow L = 1 \]

The optimal level of production is

\[ y = K^{\frac{1}{6}} L^{\frac{5}{6}} = 1^{\frac{1}{6}} 1^{\frac{5}{6}} = 1 \]

and profit

\[ \pi = 6 \times 1 - 1 \times 1 - 1 \times 1 = 4 \]

c) Secret of happiness for cost minimization is

\[ TRS = \frac{L}{K} = \frac{w_K}{w_L} = 1 \Rightarrow K = L \]

hence

\[ y = K^{\frac{1}{6}} L^{\frac{5}{6}} = K^{\frac{1}{3}} \]

hence

\[ K = L = y^3 \]

hence

\[ C(y) = 2y^3 \]
and

\[ AC = \frac{C(y)}{y} = 2y^2 \]

**Bonus Problem.** (extra 10 points)

The present value of a perpetuity is

\[
PV = \frac{x}{1 + r} + \frac{x}{(1 + r)^2} + \frac{x}{(1 + r)^3} + ... = \frac{1}{1 + r} [x + PV]
\]

Solving for \( PV \) gives

\[ PV = \frac{x}{r} \]

For the asset that pays \( x \) up to period \( T \) we have to decrease the PV by the PV of a "missing payment" after \( T \). The present value of this payment in $ from period \( T \) is \( \frac{x}{r} \) and hence in $ from period zero it is \( \left( \frac{1}{1 + r} \right)^T \frac{x}{r} \). Subtracting this number from \( PV \) for perpetuity gives

\[ PV = \frac{x}{r} - \left( \frac{1}{1 + r} \right)^T \frac{x}{r} = \frac{x}{r} \left[ 1 - \left( \frac{1}{1 + r} \right)^T \right] \]

which is the formula of the PV of annuity.