SOLOW WITH LABOR AUGMENTING TECHNOLOGICAL CHANGE

PROD. FUNCTION

TECHNOLOGICAL LEVEL

(1) $$Y = K^{\alpha} (AL)^{1-\alpha} \qquad 0<\alpha<1 \quad , A>0$$

WE CAN THINK OF "AL" AS THE "EFFECTIVE" AMOUNT OF LABOR USED IN PRODUCT.

(2) $$\hat{A} = g > 0$$ RATE OF TECHNOLOGICAL CHANGE

CONS / SAVINGS

(3) $$S = s \cdot Y \qquad 0<s<1$$

ASSUME $I = S$ SO

$Y = C + I \Rightarrow$

$C = (1-s) Y$

POPULATION = $\underbrace{\text{\# WORKERS}}_{\text{LABOR FORCE}}$ AND

(4) $$\hat{L} = n$$ RATE OF POP. GROWTH

LAW OF MOTION OF K

DEPRECIATION RATE

(5) $$\dot{K} = I - \delta K \qquad 0<\delta<1$$

USING (12) & EQUILIBRIUM CONDITION ($S=I$)

(6) $$\dot{K} = s \cdot Y - \delta K$$

SINCE $\hat{K} = \dot{K}/K \Rightarrow$

(7) $$\hat{K} = s\frac{Y}{K} - \delta$$

A BGP:

ALONG BGP, ALL VARIABLES GROW AT CONSTANT RATES, i.e:

$\hat{K}$ CONST, $\hat{Y}$ CONSTANT, ...

- FROM (17):

$$\hat{K} = s\frac{Y}{K} - \delta$$

WE HAVE THAT ALONG A BGP:

$$\frac{Y}{K} \text{ CONSTANT} \Rightarrow \boxed{\hat{Y} = \hat{K}} \quad (*)$$

- WE TAKE GROWTH RATES ON (10):

$$Y = K^{\alpha}(AL)^{1-\alpha}$$

$$\hat{Y} = \alpha \hat{K} + (1-\alpha)(\hat{A} + \hat{L})$$

USING (*) $\alpha\hat{K} = \alpha\hat{Y}$

THEN

$$\hat{Y}\cancel{(1-\alpha)} = \cancel{(1-\alpha)}(\hat{A} + \hat{L}) \Rightarrow \hat{Y} = \hat{A} + \hat{L} = g + n$$

USING (3) & (4)

SO

$$(8) \quad \boxed{\hat{Y} = g + n} \boxed{= \hat{K}}$$

NOTICE THAT THIS IMPLIES THAT ALONG A BGP: $\frac{K}{AL}$, $\frac{Y}{AL}$ ARE CONSTANT

THEN WE CAN PERFORM A CHANGE OF VARIABLES ALONG THESE LINES AND THE MODIFIED SYSTEM WILL HAVE A STEADY STATE (THAT

COINCIDES WITH THE BGP OF THE ORIGINAL VARIABLES)

2 CHANGE OF VARIABLE

NEW VARIABLES:

OUTPUT PER EFFECTIVE UNIT OF LABOR $\longrightarrow$ $\tilde{y} = \dfrac{Y}{AL} = \dfrac{y}{A}$ $\qquad \tilde{c} = \dfrac{C}{AL} = \dfrac{c}{A}$

CAPITAL PER EFFECTIVE UNIT OF LABOR $\longrightarrow$ $\tilde{k} = \dfrac{K}{AL} = \dfrac{k}{A}$

WE WILL LOOK AT THE SYSTEM IN THE MODIFIED VARIABLES $\Rightarrow$ NEED TO CALCULATE PROD. FUNCTION IN MODIFIED VARIABLES AND LAW OF MOTION OF $\tilde{k}$.

(i) PROD. FUNCTION.

DIVIDE (10) BY AL

$$\frac{Y}{AL} = \frac{K^{\alpha}(AL)^{1-\alpha}}{AL} = \frac{K^{\alpha}\cancel{(AL)^{1-\alpha}}}{(AL)^{\alpha}\cancel{(AL)^{1-\alpha}}} = \left(\frac{K}{AL}\right)^{\alpha}$$

i.e.

$$(10) \qquad \tilde{y} = \tilde{k}^{\alpha}$$

(ii) LAW OF MOTION OF $\tilde{k}$

SINCE $\tilde{k} = \dfrac{K}{AL}$ USING HAT ALGEBRA:

$$\hat{\tilde{k}} = \hat{K} - \hat{A} - \hat{L} = \hat{K} - (g+n)$$

USING (7)

$$\hat{\tilde{k}} = s\frac{Y}{K} - \delta - (g+n) \underset{\substack{\text{DIVIDING/MULT.}\\ \text{1ST TERM BY AL}}}{=} s\frac{Y/AL}{K/AL} - (\delta+g+n)$$

$$\hat{\tilde{k}} = \alpha \frac{\tilde{y}}{\tilde{k}} - (\delta + g + n)$$

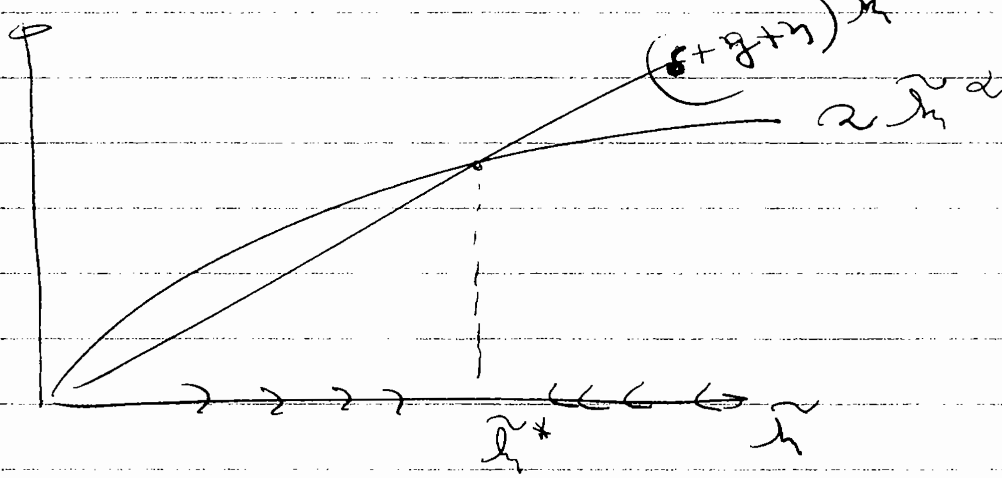
USING (10)

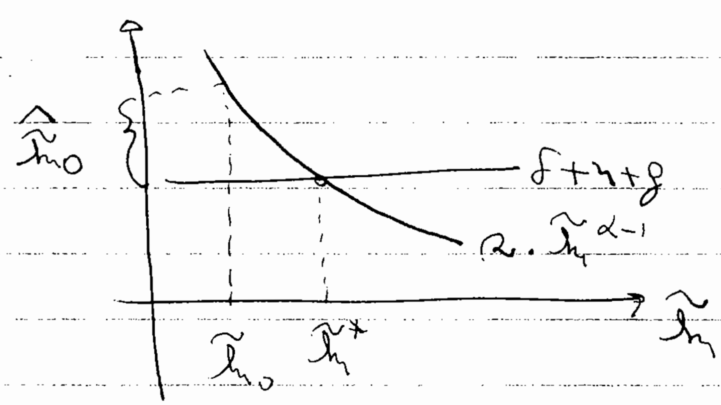
(11) $$\boxed{\hat{\tilde{k}}} = \alpha \frac{\tilde{k}^{\alpha}}{\tilde{k}} - (\delta+g+n) = \boxed{\alpha \tilde{k}^{\alpha-1} - (\delta+g+n)}$$

SINCE $\hat{\tilde{k}} = \dot{\tilde{k}} / \tilde{k} \Rightarrow \dot{\tilde{k}} = \hat{\tilde{k}} \cdot \tilde{k}$

AND

(12) $$\boxed{\dot{\tilde{k}} = \alpha \tilde{k}^{\alpha} - (\delta+g+n)\tilde{k}}$$ SOLOW EQUATION





NOTICE:
$\tilde{k}^*$ IS THE S.S.
AND
$\hat{\tilde{k}}^* = 0$

ANALYSIS OF SYSTEM

THE MODIFIED SYSTEM HAS A S.S. AT $\tilde{k}^*$

SINCE $\hat{\tilde{k}}^* = 0$

I

IF $\tilde{k} < \tilde{k}^* \Rightarrow \hat{\tilde{k}} > 0$ & $\tilde{k} \uparrow$

IF $\tilde{k} > \tilde{k}^* \Rightarrow \hat{\tilde{k}} < 0$ & $\tilde{k} \downarrow$

i.e. NO MATTER WHERE WE START WE END UP AT S.S.

SINCE $\hat{\tilde{y}} = \alpha \hat{\tilde{k}}$ (BY TAKING GROWTH RATES OF (10))

WE HAVE

$$\hat{\tilde{y}}^* = 0 \quad \text{AT S.S.}$$

IF $\tilde{y} < \tilde{y}^* \Rightarrow \hat{\tilde{y}} > 0 \Rightarrow \tilde{y} \uparrow$

IF $\tilde{y} > \tilde{y}^* \Rightarrow \hat{\tilde{y}} < 0 \Rightarrow \tilde{y} \downarrow$

NOTICE THAT THE S.S. OF MODIFIED SYSTEM IS THE BGP OF ORIGINAL ONE –

AT S.S. $\hat{\tilde{y}} = \widehat{\left(\frac{Y}{AL}\right)} = 0 \Rightarrow \widehat{\left(\frac{Y}{L}\right)} = \hat{A}$

OR $\boxed{\hat{y}} = \underset{g}{\overset{\|}{\hat{A}}} = \boxed{g}$

SO OUTPUT PER WORKER GROWS AT THE RATE OF TECH. CHANGE

II

LEVELS

AT S.S. $\hat{\tilde{k}} = 0 \Rightarrow$ BY (11) THAT $0 = s\tilde{k}^{*\alpha-1} - (\delta + g + n)$

$$\Rightarrow s\tilde{k}^{*\alpha-1} = \delta + g + n$$

$$\left(\frac{s}{\delta+g+n}\right)^{\frac{1}{1-\alpha}} = \tilde{k}^* \underset{\text{using }(10)}{\Longrightarrow} \boxed{\tilde{y}^* = \left(\frac{s}{\delta+g+n}\right)^{\frac{\alpha}{1-\alpha}}} \quad (13)$$

SINCE $\tilde{y}^* = \dfrac{y^*_t}{A_t} \Rightarrow y^*_t = A_t \tilde{y}^*$

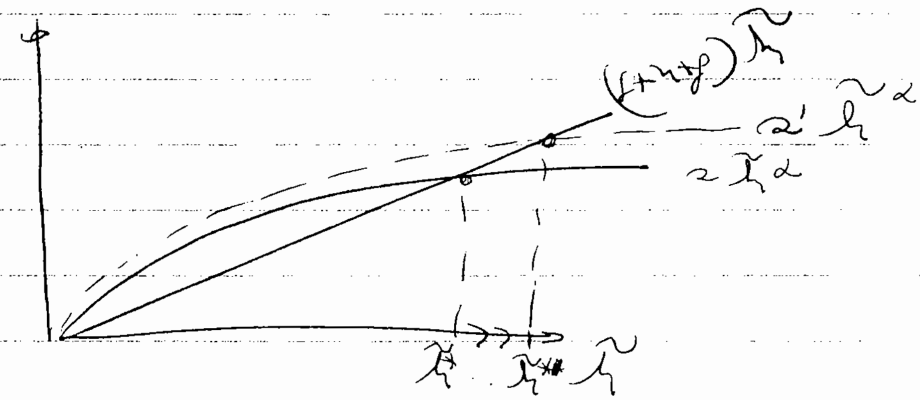
i.e. USING (13)

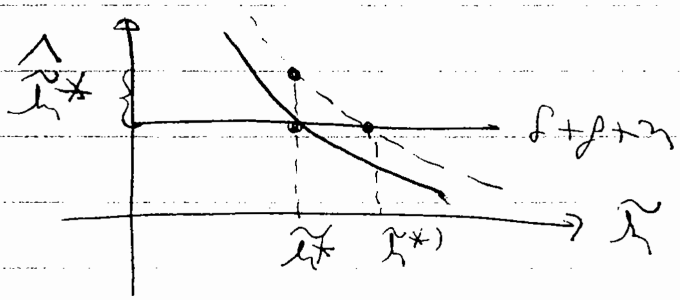
$$\boxed{y^*_t = \left(\frac{s}{\delta+g+n}\right)^{\frac{\alpha}{1-\alpha}} A_t}$$

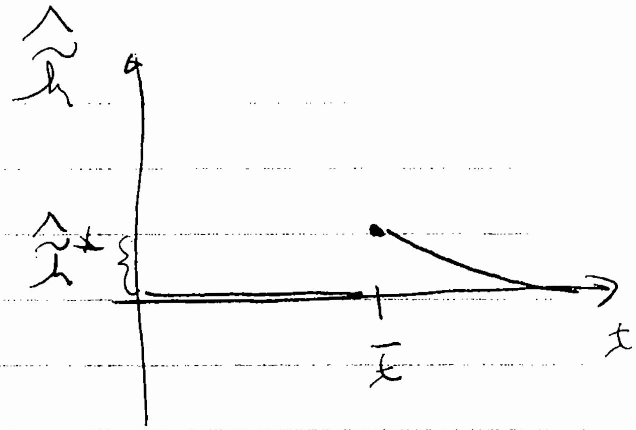
ALONG A BGP OUTPUT PER WORKER FOLLOWS THIS PATH

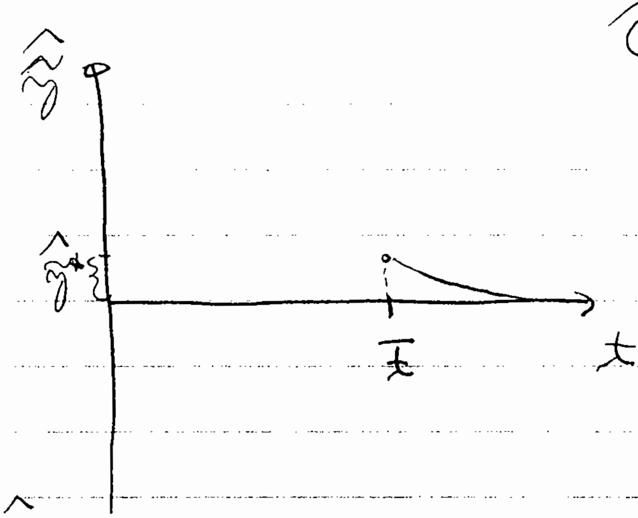
COMPARATIVE STATICS (EFFECT ON PATH OF OUTPUT PER WORKER)

(1) $\uparrow s$: EFFECTS ON $\hat{y}_t$ & y_t

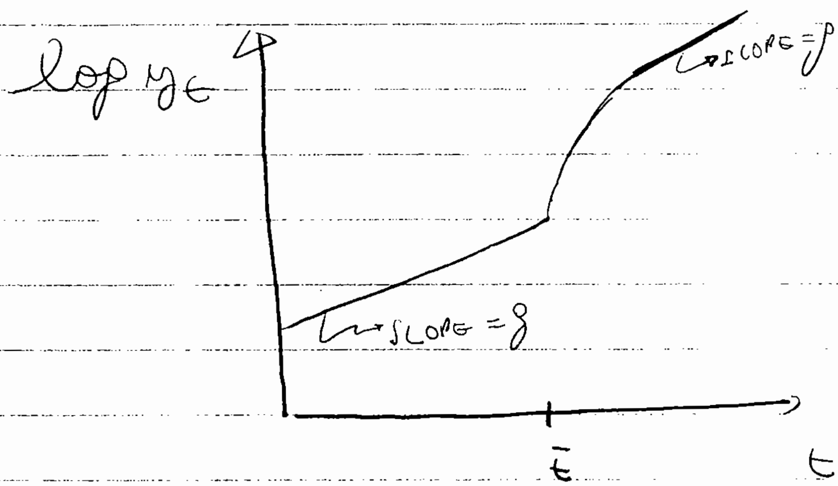


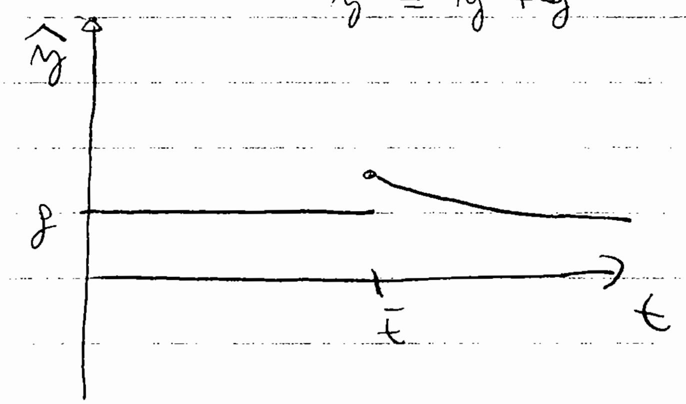






SINCE $\hat{\tilde{y}} = \alpha \hat{\tilde{k}}$ AND $\tilde{y} = y/A \Rightarrow$
$\hat{y} = \hat{\tilde{y}} + g$





NOTICE SLOPE OF $\log y_t$
BIGGER THAN g DURING
ADJUSTMENT PERIOD