MATH NEEDED

VARIABLES ARE FUNCTIONS OF TIME

EX: $X(t)$, $Y(t)$, $y(t)$

HOW DO THE VARIABLES CHANGE OVER TIME?

$$\frac{dX(t)}{dt}, \quad \frac{dY(t)}{dt}, \quad \frac{dy(t)}{dt}$$

WE USE THE "DOT" NOTATION TO REPRESENT THEM, i.e.

$$\dot{X} = \frac{dX(t)}{dt}, \quad \dot{Y} = \frac{dY(t)}{dt}, \quad \dot{y} = \frac{dy(t)}{dt}.$$

NOTICE:

$$\dot{X} = \underbrace{\frac{dX(t)}{dt}}_{\text{INSTANTANEOUS CHANGE}} = \lim_{\Delta t \to 0} \frac{X(t) - X(t - \Delta t)}{\Delta t}$$

GROWTH RATES. WE USE THE "HAT" NOTATION

$$\hat{X} = \frac{\dot{X}}{X} = \frac{dX(t)}{dt} \cdot \frac{1}{X(t)}$$

THEN $\dot{X} = \hat{X} \cdot X$

NOTICE:

$$\frac{dX(t)}{dt} \cdot \frac{1}{X(t)} = \frac{d \log X(t)}{dt}$$

SO THE GROWTH RATE IS THE DERIVATIVE OF THE LOG.

REMARK 1:

IF $\dot{X} = 0$ (i.e. $\hat{X} = 0$) $\Rightarrow$ THE VARIABLE X IS NOT CHANGING (IT IS CONSTANT)

RULES: HAT ALGEBRA

$$\widehat{(XY)} = \hat{X} + \hat{Y}$$

$$\widehat{(X/Y)} = \hat{X} - \hat{Y}$$

$$\widehat{(X^{\alpha})} = \alpha \hat{X}$$

STEADY STATE OF A SYSTEM/VARIABLE

SITUATION WHERE THE VALUE OF THE VARIABLES/VARIABLE DOES NOT CHANGE

WE USE THE NOTATION: S.S. FOR STEADY STATE

EX: SYSTEM OF VARIABLES $X(t)$ & $Y(t)$

AT S.S. $\hat{X} = 0$ $\hat{Y} = 0$

(ALSO $\dot{X} = 0$ $\dot{Y} = 0$)

BALANCED GROWTH PATH:
(OF SYSTEM OR
VARIABLE)
SITUATION WHERE ALL VARIABLES
GROW THAT CONSTANT RATES
i.e:

$$\hat{Y} = \text{CONSTANT}$$
$$\hat{X} = \text{"}$$

) NOT NECESSARILY THE SAME CONSTANT

DIAGRAMS - HELP WITH
PATHS OF VARIABLES
AND GROWTH
RATES

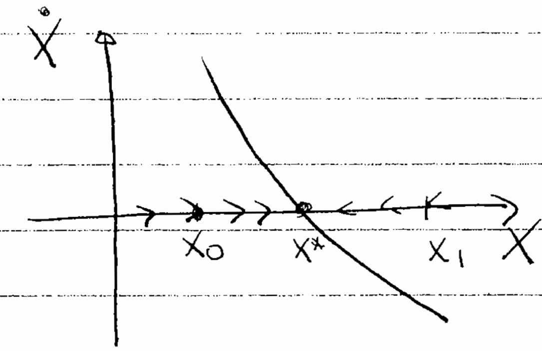
EX: IF $\Delta Y > 0 \Rightarrow Y \uparrow$
$(\text{i.e. } \hat{Y} > 0)$

IF $\Delta Y = 0 \Rightarrow Y$ CONSTANT

IF $\Delta Y < 0 \Rightarrow Y \downarrow$

EXAMPLE OF DIAGRAMS WE WILL BE USING

EX 1:



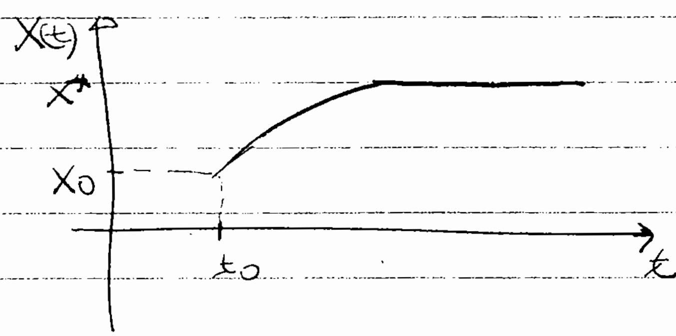
ASSUME WE HAVE THE FUNCTION AS SHOWN

AT X^*, $\dot{X}=0$ (ALSO $\hat{X}=0$) $\Rightarrow$ STEADY STATE AT X^*

IF $X < X^* \Rightarrow \dot{X} > 0$ (ALSO $\hat{X} > 0$) $\Rightarrow X \uparrow$

IF $X > X^* \Rightarrow \dot{X} < 0$ (ALSO $\hat{X} < 0$) $\Rightarrow X \downarrow$

IF $X = X_0$ AT TIME $t = t_0$ THEN WE HAVE THE FOLLOWING PATH FOR $X(t)$



EXPONENTIAL FUNCTION

$$X(t) = C \cdot e^{gt}$$

$C > 0$ IS A CONSTA

TAKING LOGS

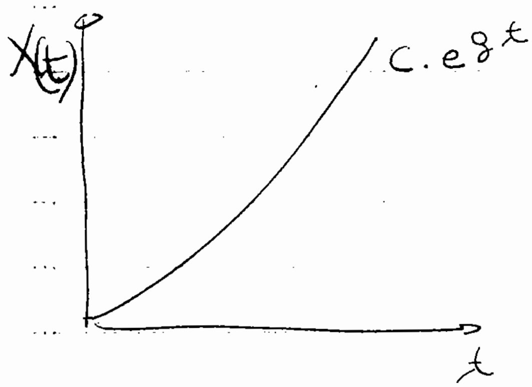
$$\log X(t) = \log C + gt$$

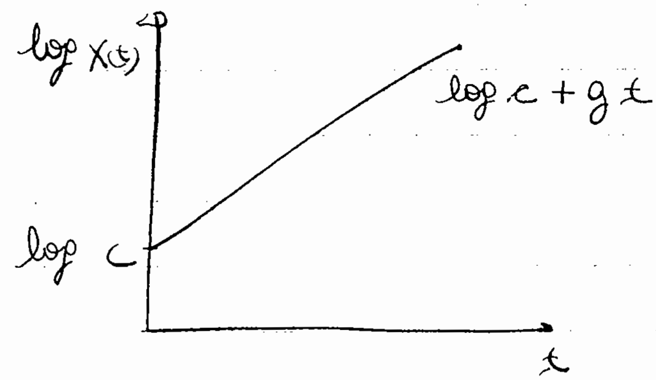
TAKING DERIVATIVES

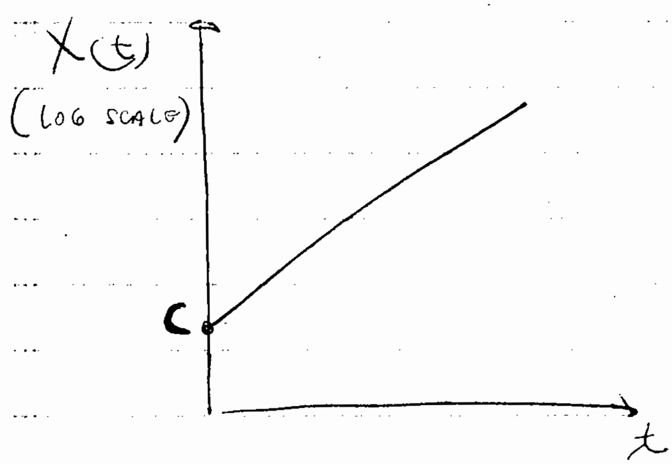
$$\frac{d \log X(t)}{dt} = g$$

WE CAN THEN WRITE THE GROWTH RATE OF X(

$$\hat{X} = \frac{\dot{X}}{X} = \frac{d \log X(t)}{dt} = g$$







TECHNOLOGY / PROD FUNCTION REVIEW

- PROD. FUNCTION
  - MARG. PRODUCTS
  - RETURNS TO SCALE

PROD. FUNCTION : 2 INPUTS (K, L)

$$Y = F(K, L) = A\, K^{\alpha} L^{1-\alpha}$$

$$A > 0, \quad 0 < \alpha < 1$$

MPL = MARGINAL PRODUCT LABOR = $\left.\dfrac{\Delta Y}{\Delta L}\right|_{\text{SMALL CHANGES}}$

MPK = " " CAPITAL = $\left.\dfrac{\Delta Y}{\Delta K}\right|_{\text{SMALL CHANGES}}$

IN THIS CASE

$$\left.\begin{array}{l} MPL \downarrow \text{ AS } L \uparrow \\ MPK \downarrow \text{ AS } K \uparrow \end{array}\right] \text{ i.e. } \downarrow \text{ MARGINAL PRODUCTS}$$

RETURNS TO SCALE

THIS FUNCTION HAS CONSTANT RETURNS TO SCALE (CRS) i.e.

IF K & L CHANGE IN THE SAME PROPORTION, Y CHANGES IN THE SAME PROPORTION

EXAMPLE: ASSUME K & L $\uparrow$ BY 30%

$$\Rightarrow K_1 = 1.30\, K_0$$

$$L_1 = 1.30\, L_0$$

②

THEN $Y_1 = F(K_1, L_1) = A \cdot (1.30 K_0)^{\alpha} (1.30 L_0)^{1-\alpha}$

$$= A \, (1.30)^{\alpha + 1 - \alpha} K_0^{\alpha} L_0^{1-\alpha}$$

$$= 1.30 \underbrace{A K_0^{\alpha} L_0^{1-\alpha}}_{Y_0} = 1.30 \, Y_0$$

COBB DOUGLASS PRODUCTION FUNCTION

& FACTOR SHARES $\quad Y = F(K,L) = A K^{\alpha} L^{1-\alpha}$

WE WILL SHOW THAT

RENTAL PRICE OF K

α = CAPITAL SHARE IN OUTPUT $= \dfrac{r \cdot K}{p \cdot Y}$

WAGE

$1-\alpha$ = LABOR SHARE IN OUTPUT $= \dfrac{w \cdot L}{p \cdot Y}$

STEP 1: CALCULATE MPL, MPK

STEP 2: USE PROFIT MAX. CONDITIONS & STEP 1 TO GET RESULT.

STEP 1: MARGINAL PRODUCT CALCULATIONS

$$MPL = \frac{\partial Y}{\partial L} = A\; K^{\alpha}\, (1-\alpha)\, L^{1-\alpha-1} = (1-\alpha)\frac{A K^{\alpha} L^{1-\alpha}}{L}$$

$$= (1-\alpha)\frac{Y}{L} \quad \Rightarrow \boxed{(1-\alpha) = \frac{MPL \cdot L}{Y}} \quad (*)$$

$$MPK = \frac{\partial Y}{\partial K} = A\,\alpha K^{\alpha-1} L^{1-\alpha} = \alpha \frac{A K^{\alpha} L^{1-\alpha}}{K} = \alpha \frac{Y}{K}$$

$$\Rightarrow \boxed{\alpha = \frac{MPK \cdot K}{Y}} \quad (**)$$

STEP 2:

PROFIT MAXIMIZATION $\Rightarrow$ $W = MPL \cdot p$

$r = MPK \cdot p$

THEN $MPL = W/p$

$MPK = r/p$

USING THE PREVIOUS EQUATIONS & (*) (**)

WE HAVE $\boxed{(1-\alpha) = \dfrac{wL}{pY}}$ $\quad \boxed{\alpha = \dfrac{rK}{pY}}$

CHANGE OF VARIABLE

$Y = F(K, L)$ (3 VARIABLES)

$= A K^{\alpha} L^{1-\alpha}$

DIFFICULT TO GRAPH, ETC

WE WILL MAKE A CHANGE OF VARIABLE TO HAVE ONLY TWO

$\Rightarrow$ EASY DIAGRAM.

NEW VARIABLES: PER PERSON/WORKER

WE CAN DO THIS BECAUSE OUR FUNCTION HAS CRS

STEPS

DIVIDE BOTH SIDES BY L

$$\frac{Y}{L} = \frac{A K^{\alpha} L^{1-\alpha}}{L}$$

WORK WITH RIGHT HAND SIDE

$$\frac{Y}{L} = A K^{\alpha} L^{1-\alpha} \cancel{L^{-1}} = A \frac{K^{\alpha}}{L^{\alpha}} = A\left(\frac{K}{L}\right)^{\alpha}$$

$$\boxed{y = A k^{\alpha}}$$

SO OUR NEW VARIABLES ARE:

$y = Y/L$, $k = K/L$

OUTPUT PER PERSON/WORKER — CAPITAL PER PERSON/WORKER

SINCE $0 < \alpha < 1$ $\Rightarrow$

$\uparrow k \Rightarrow \uparrow y$

BUT AT $\downarrow$ RATE

2