5-28-09

- LAST CLASS:

SOLOW WITHOUT TECHNOLOGICAL CHANGE

(ORIGINAL VARIABLES

- TODAY: MODIFIED SYSTEM (SO THAT IT HAS A STEADY STATE)

- DESCRIBE BGP/S.S.
- " BEHAVIOR OUTSIDE S.S.
- GOLDEN RULE

LAST CLASS: ORIGINAL SYSTEM

$$Y = F(K, L) = K^\alpha L^{1-\alpha}$$

$\hat{L} = n$

$$\dot{K} = \alpha Y - \delta K$$

$$\hat{K} = \alpha \frac{Y}{K} - \delta$$

AT BGP: $\hat{K} = \hat{Y} = \hat{L} = n$

MODIFIED SYSTEM: REMARK: PARAMETERS: $\alpha, \delta, \alpha, n$
VARIABLES: $y, k$

HOLD ALWAYS

$$\underbrace{\frac{Y}{L}}_{y} = \frac{F(K, L)}{L} = \frac{K^\alpha L^{1-\alpha}}{L} = \Big(\underbrace{\frac{K}{L}}_{k}\Big)^\alpha$$

$$\Rightarrow \boxed{y = k^\alpha} \quad (10)$$

$$\boxed{\hat{k}} = \hat{K} - \hat{L} = \alpha \frac{Y/L}{K/L} - \delta - n = \boxed{\alpha \frac{y}{k} - (\delta + n)} \quad (12)$$

$$\Rightarrow \hat{k} = \boxed{\alpha k^{\alpha-1} - (\delta+n)} \quad (12')$$

then $\boxed{\dot{k}} = \hat{k} \cdot k = \boxed{\alpha k^\alpha - (\delta+n) k}$ (14)

AT BGP $\hat{k}, \hat{y}$, etc ~~should~~ ARE CONSTANT

THEN (12) $\Rightarrow \frac{y}{k}$ CONSTANT $\Rightarrow \boxed{\hat{y} = \hat{k}}$ ①

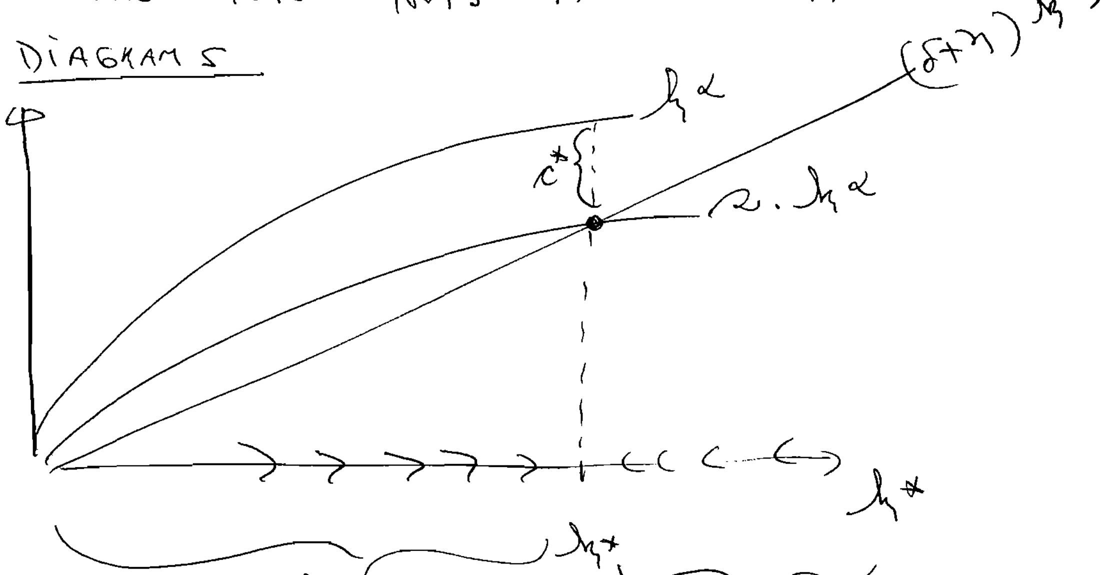
TAKING GROWTH RATES ON (10)

$$\boxed{\hat{y} = \alpha \hat{k}} \quad (\Delta\Delta)$$

THEREFORE USING (Δ) WE & (ΔΔ) GET AT BGP OR S.S.: $\boxed{\hat{k} = \hat{y} = 0}$ ⇒ $y$ & $k$ CONSTANT

SO THE SYSTEM HAS A STEADY STATE. (S.S.)

DIAGRAMS




$\dot{k} > 0 \Rightarrow k\uparrow$

$\dot{k} < 0 \Rightarrow k\downarrow$

## ANALYSIS OF BGP/S.S. IN MODIFIED SYSTEM & OUT OF S.S. BEHAVIOR

(1) SYSTEM HAS A S.S. AT $k^*$

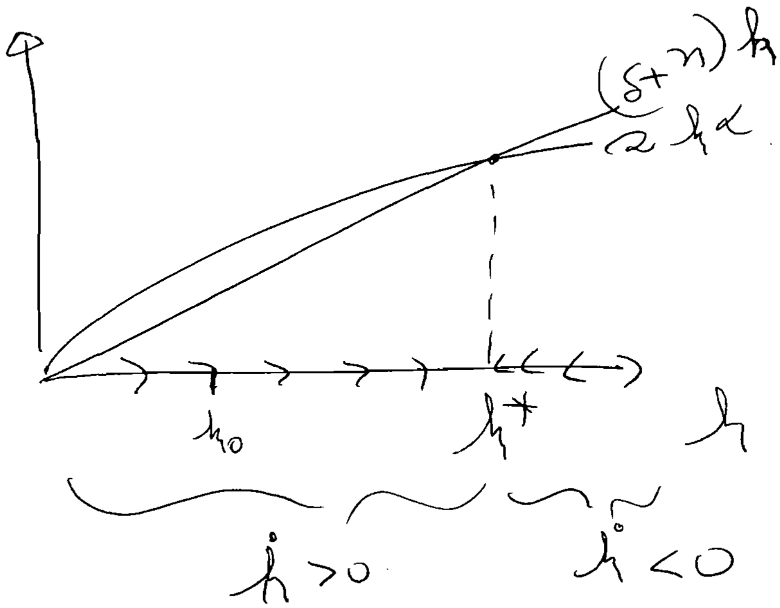
WHERE $\hat{k} = 0$, $\dot{k} = 0$

IF $k < k^*$ : $s k^\alpha > (\delta+n) k$

$\Rightarrow$ BY (14) THAT $\dot{k} > 0$ i.e. $k\uparrow$

IF $k > k^*$ : $s k^\alpha < (\delta+n) k$

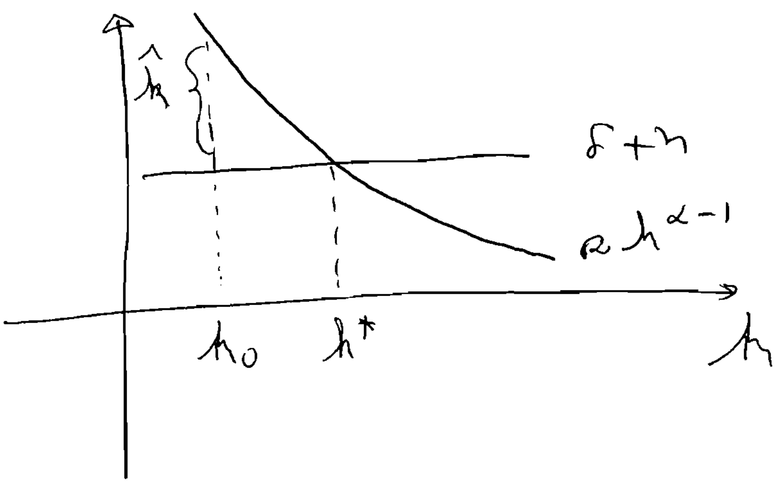
$\Rightarrow$ BY (14) THAT $\dot{k} < 0$ i.e. $k\downarrow$

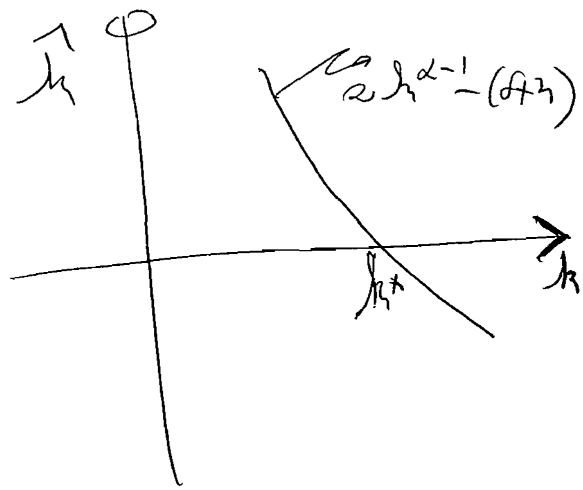


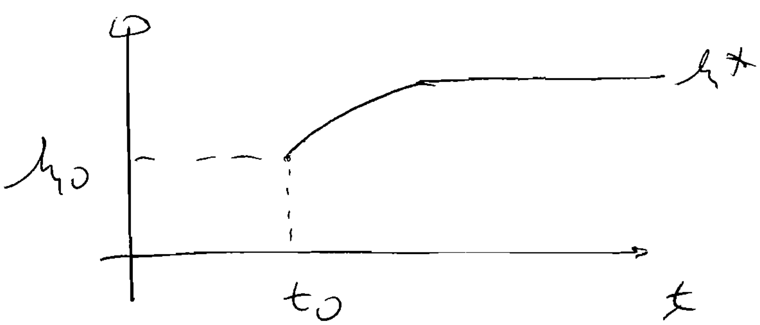

(2) PATHS, GROWTH RATES, ETC

WE REWRITE RELEVANT EQUATIONS & DRAW FUNCTIONS

$$\hat{k} = a k^{\alpha-1} - (\delta+n)$$
$$\dot{k} = a k^{\alpha} - (\delta+n)k$$




SINCE

$\hat{y} = \alpha \hat{k}$

AT S.S. $\hat{y} = 0$

AND PATHS JUST DIFFER BY A CONSTANT $\alpha$

(3) AT S.S. OF MODIFIED SYSTEM (OR BGP OF ORIGINAL):

$$\hat{y} = \widehat{\left(\frac{Y}{L}\right)} = 0 \implies \hat{Y} = \hat{L} = n$$

$$\hat{k} = \widehat{\left(\frac{K}{L}\right)} = 0 \implies \hat{K} = \hat{L} = n$$

(4) CALCULATION OF S.S. $k^*$, $y^*$, $c^*$

AT S.S. : $\hat{k} = s k^{\alpha-1} - (\delta+n) = 0$

USING (12') $\implies s k^{\alpha-1} = \delta+n \implies \frac{s}{\delta+n} = k^{1-\alpha}$

$$\implies \boxed{k^* = \left(\frac{s}{\delta+n}\right)^{\frac{1}{1-\alpha}}} \quad (15)$$

THEN

$$(16) \quad \boxed{y^* = k^{*\alpha} = \left(\frac{s}{\delta+n}\right)^{\frac{\alpha}{1-\alpha}}} \quad \&$$

$$(17) \quad \boxed{c^* = (1-s)\left(\frac{s}{\delta+n}\right)^{\frac{\alpha}{1-\alpha}}}$$

REMARKS / RESULTS

I LEVEL RESULTS : OUTPUT PER WORKER AT BGP/S.S.

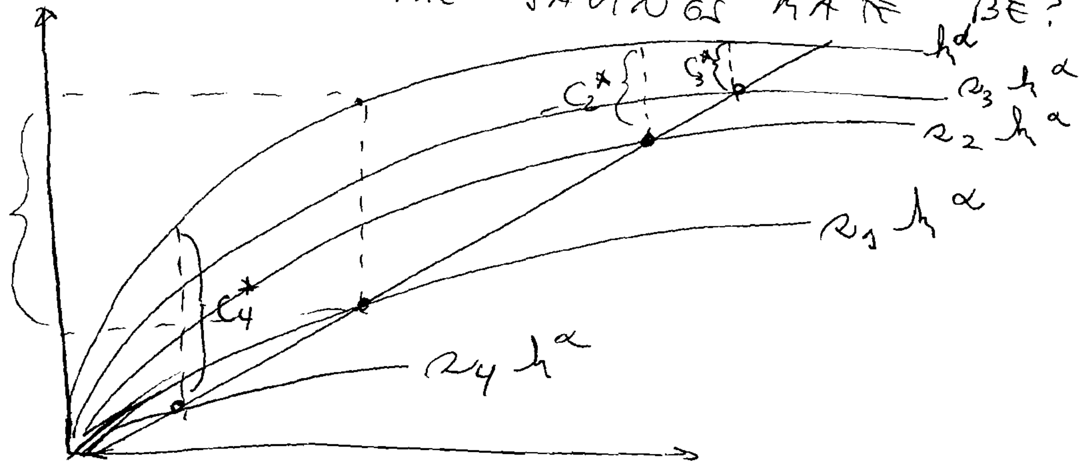
$\boxed{y^* \quad \uparrow}$ WITH INCREASES IN SAVINGS RATE
DECREASES IN RATE OF POP. GROWTH
DECREASES IN DEPRECIATION RATE

II GROWTH RATES

- AT BGP/S.S. THE GROWTH RATE OF OUTPUT PER WORKER IS ZERO
- IF CAPITAL PER WORKER $< k^*$ (S.S. VALUE) $\Longrightarrow$ GROWTH RATE OF OUTPUT PER WORKER IS POSITIVE
- IF CAPITAL PER WORKER $> k^*$ (S.S. VALUE) $\Longrightarrow$ GROWTH RATE OF OUTPUT PER WORKER IS NEGATIVE
- THE FURTHER THE CAPITAL PER WORKER IS FROM THE S.S. THE HIGHER THE GROWTH RATE OF $k$ AND $y$ IN ABSOLUTE VALUE

## GOLDEN RULE

ASSUME GOAL IS TO MAXIMIZE CONSUMPTION PER CAPITA/WORKER AT THE STEADY STATE. WHAT SHOULD THE SAVINGS RATE BE?




WE SEE IN DIAGRAM THAT HIGHER SAVINGS RATES DO NOT NECESSARILY RESULT IN HIGHER CONSUMPTION PER WORKER AT THE S.S. (ALTHOUGH THEY RESULT IN HIGHER PER CAPITA OUTPUT AT S.S.)

GOLDEN RULE SAVINGS RATE: $s_G$

MAX. S.S. CONS. PER WORKER:

USING EQ. (17)

$$\max_s c^* = \max_s (1-s)\left(\frac{s}{\delta+n}\right)^{\frac{\alpha}{1-\alpha}}$$

$$= \max_s (1-s)\, s^{\frac{\alpha}{1-\alpha}} \left(\frac{1}{\delta+n}\right)^{\frac{\alpha}{1-\alpha}}$$

$$\frac{\partial}{\partial s} = \left[(-1)\, s^{\frac{\alpha}{1-\alpha}} + \frac{\alpha}{1-\alpha}\, s^{\frac{\alpha}{1-\alpha}-1}(1-s)\right]\left[\frac{1}{\delta+n}\right]^{\frac{\alpha}{1-\alpha}} = 0$$

$$\Rightarrow (-1) + \frac{\alpha}{1-\alpha}(1-s)\,s^{-1} = 0 \Rightarrow \frac{\alpha}{1-\alpha} = \frac{s}{1-s}$$

$$\Rightarrow \boxed{s^G = \alpha}$$

ALTERNATIVE METHOD

LOOKING AT DIAGRAM, $c^*$ is MAXIMIZED WHEN: SLOPE OF $f(k)$ = SLOPE OF $(\delta+n)k$

$$\Rightarrow f'(k) = \delta + n$$

$$\alpha k^{\alpha-1} = \delta + n \Rightarrow \boxed{k^{G\,1-\alpha} = \frac{\alpha}{\delta+n}} \quad \triangle$$

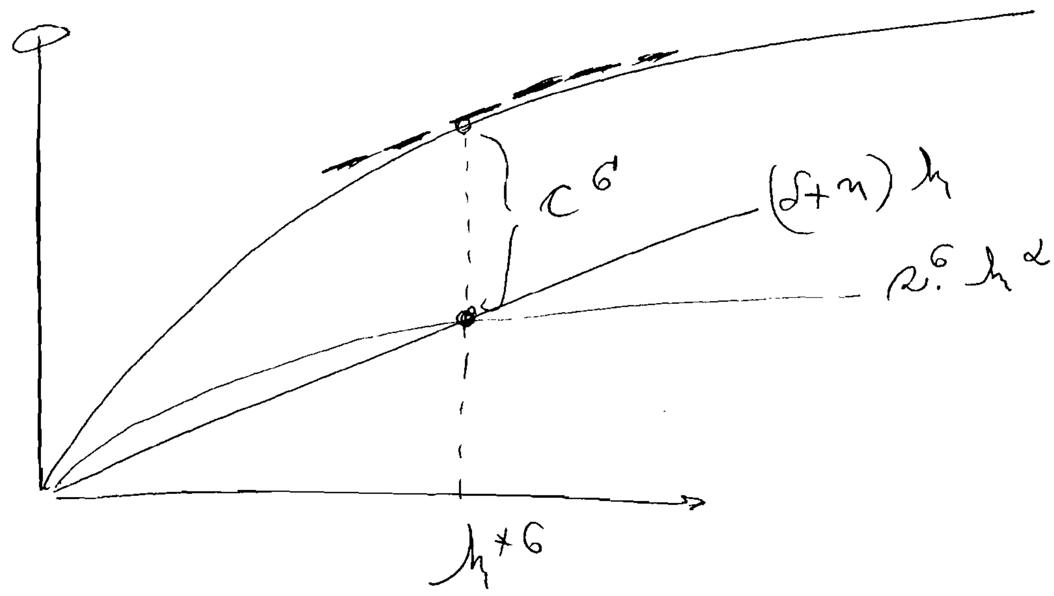
THEN WE USE (15) TO FIGURE OUT $s^G$

(15) CAPITAL PER WORKER AT S.S.

$$k^* = \left[\frac{s}{\delta+n}\right]^{\frac{1}{1-\alpha}} \Rightarrow \boxed{k^{*\,1-\alpha} = \frac{s}{\delta+n}} \quad \triangle\triangle$$

SINCE WE NEED $k^G = k^*$ USING $\triangle$ & $\triangle\triangle$ WE CONCLUDE THAT

$$\boxed{s^G = \alpha}$$


c
(δ+n) k
k*