5-26-09

# MATH REVIEW +++

VARIABLES ARE FUNCTIONS OF TIME $t$

EX: $X(t)$ OR $X_t$

$Y(t)$

$y(t)$

LOOKING AT HOW VARIABLES CHANGE OVER TIME:

$$\frac{d\,X(t)}{dt}, \quad \frac{d\,Y(t)}{dt}, \quad \frac{d\,y(t)}{dt}$$

WE WILL USE "DOT NOTATION"

$$\dot{X} = \frac{d\,X(t)}{dt}, \quad \dot{Y} = \frac{d\,Y(t)}{dt}$$

$$\dot{y} = \frac{d\,y(t)}{dt}$$

**NOTICE:**

$$\dot{X} = \frac{d\,X(t)}{dt} = \cancel{\lim_{\Delta t \to 0} X(t)-X(t-1)}$$

$$= \lim_{\Delta t \to 0} \frac{X(t)-X(t-\Delta t)}{\Delta t}$$

## GROWTH RATES

USE "HAT NOTATION"

$$\hat{X} = \frac{\dot{X}}{X} = \frac{d\,X(t)}{d\,t} \cdot \frac{1}{X(t)}$$

$$\Longrightarrow \dot{X} = \hat{X} \cdot X$$

NOTICE: THE GROWTH RATE IS THE DERIVATIVE OF THE LOG.

$$\hat{X} = \frac{d\,X(t)}{d\,t} \cdot \frac{1}{X(t)} = \frac{d \log X(t)}{d\,t}$$

## RULES OF "HAT ALGEBRA"

PRODUCT:

$$\widehat{(X \cdot Y)} = \hat{X} + \hat{Y}$$

QUOTIENT

$$\widehat{\left(\frac{X}{Y}\right)} = \hat{X} - \hat{Y}$$

EXPONENT

$\alpha$ CONSTANT POSITIVE

$$\widehat{(X^{\alpha})} = \alpha \cdot \hat{X}$$

## STEADY STATE OF A SYSTEM

SITUATION WHERE THE VALUE OF A VARIABLE/VARIABLES DOES NOT CHANGE.

WE WILL USE THE NOTATION:

S.S. (STEADY STATE OF A SYSTEM)

EX: SYSTEM WITH 2 VARIABLES: $X(t)$, $Y(t)$

$\Longrightarrow$ AT S.S: $\dot{X}=0$, $\dot{Y}=0$

AND $\hat{X}=0$, $\hat{Y}=0$

## BALANCED GROWTH PATH OF A SYSTEM

SITUATION WHERE ALL VARIABLES GROW AT CONSTANT RATES.

EX:

$\hat{X}$ = CONSTANT

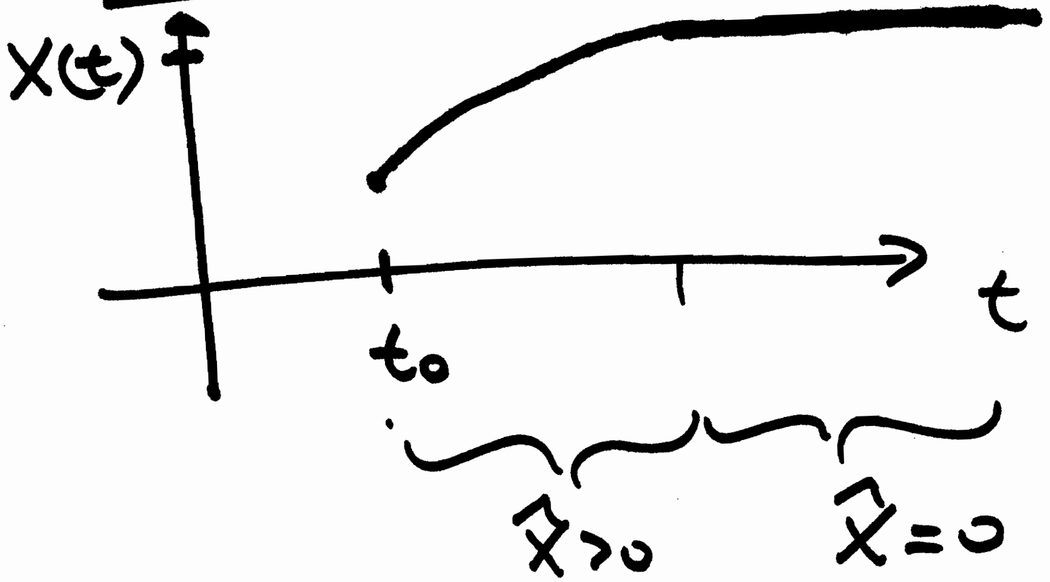
$\hat{Y}$ = CONSTANT

IF $\hat{Y} > 0 \Rightarrow Y \uparrow$

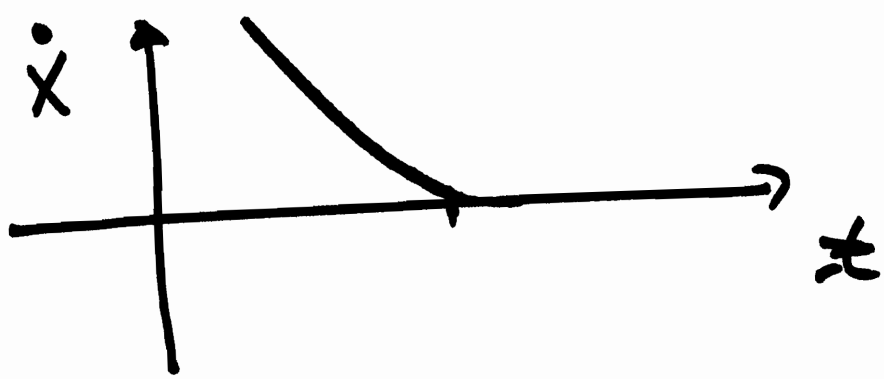
$\hat{Y} = 0 \Rightarrow Y$ CONSTANT

$\hat{Y} < 0 \Rightarrow Y \downarrow$

## EXAMPLES OF DIAGRAMS




EX:



# EXAMPLE EXPONENTIAL

$$X(t) = c \cdot e^{gt}$$

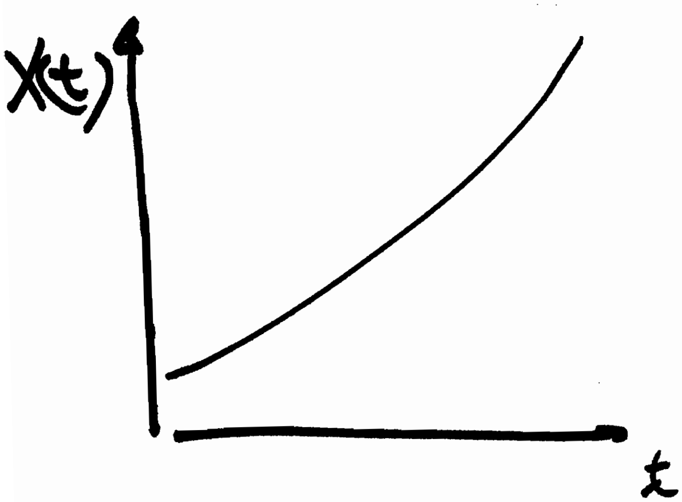
WHERE

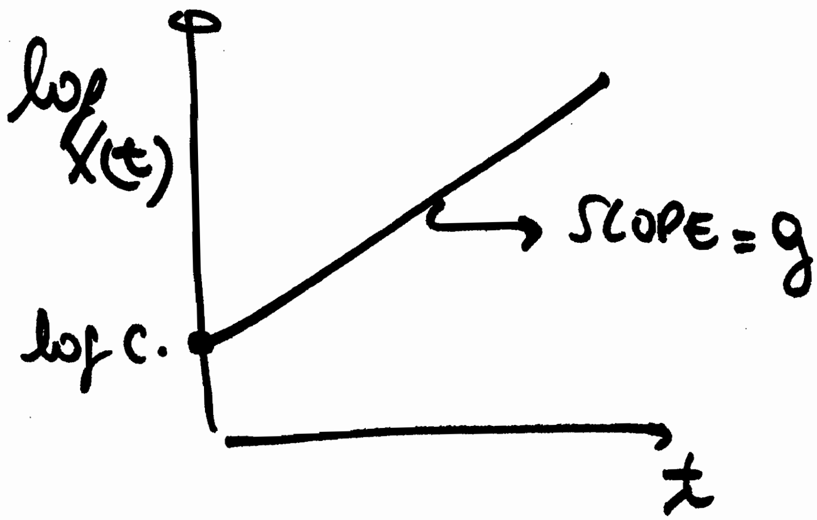
$c$, $g$ POSITIVE CONSTANTS

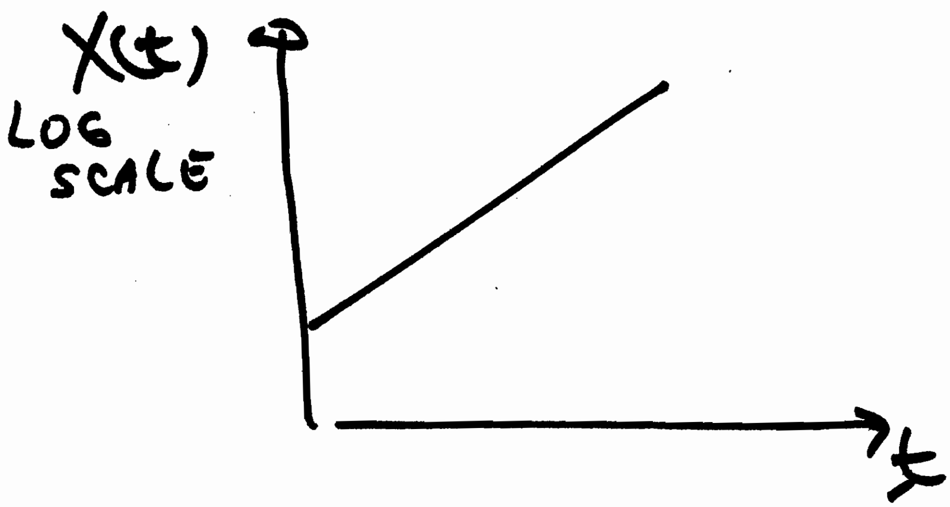
TAKING LOGS

$$\log X(t) = \log c + g\,t$$

$$\frac{d \log X(t)}{d t} = g \boxed{= \hat{X}}$$



# TECHNOLOGY / PROD. FUNCTION REVIEW

MARGINAL PRODUCTS
RETURNS TO SCALE
PROFIT MAXIMIZATION

EXAMPLE: 2 INPUTS: $L, k$

$$Y = F(k, L) = A \cdot k^{\alpha} L^{1-\alpha}$$

$$A > 0, \quad 0 < \alpha < 1$$

$$MPL = \frac{\Delta Y}{\Delta L}\Bigg|_{\text{SMALL CHANGES}} = \frac{\partial F(\,)}{\partial L} =$$

$$= A\, k^{\alpha} (1-\alpha)\, L^{\cancel{1}-\alpha-\cancel{1}}$$

$$= A\,(1-\alpha)\; k^{\alpha}\, L^{-\alpha}$$

$$= \boxed{A\,(1-\alpha)\left(\frac{k}{L}\right)^{\alpha}} \quad \triangle 1$$

$$MPK = \frac{\Delta Y}{\Delta k}\Bigg|_{\text{SMALL CHANGES}} = \frac{\partial F(\,)}{\partial k} = \quad \triangle 2$$

$$= A\, L^{1-\alpha}\, \alpha\, k^{\alpha-1} = \boxed{A\,\alpha\left(\frac{L}{k}\right)^{1-\alpha}}$$

# FACTOR SHARES & PROD. FUNCTION COEFFICIENTS

WE WILL SHOW THAT

$$\alpha = \text{CAPITAL SHARE IN OUTPUT} = \frac{r \cdot k}{p \cdot Y}$$

WHERE $r$ = RENTAL PRICE OF CAPITAL

$p$ = OUTPUT PRICE

$$1-\alpha = \text{LABOR SHARE IN OUTPUT} = \frac{W \cdot L}{p \cdot Y}$$

WHERE $W$ = WAGE

WE NEED TO ASSUME PERFECT COMPETITION AND PROFIT MAXIMIZATION.

PERFECT COMPETITION + PROFIT MAX

$\Longrightarrow$ INPUTS ARE PAID VALUE OF THEIR MARGINAL PRODUCT

(*)
$$\boxed{\begin{aligned} W &= MPL \cdot p \\ r &= MPk \cdot p. \end{aligned}}$$

WORKING AGAIN WITH OUR MARGINAL PRODUCTS.

USING $\triangle 1$

$$MPL = A(1-\alpha)\left(\frac{K}{L}\right)^{\alpha}$$

$$= A(1-\alpha)K^{\alpha}L^{-\alpha}$$

MULTIPLY & DIVIDE RIGHT HAND SIDE BY $L$

$$= \frac{(1-\alpha)}{L}\underbrace{A K^{\alpha} L^{1-\alpha}}_{Y}$$

$$MPL = (1-\alpha)\frac{Y}{L}$$

MULTIPLY BOTH SIDES BY $p$:

$$\underbrace{p.MPL}_{=W \text{ BY } (*)} = (1-\alpha)\frac{p.Y}{L}$$

$$\Rightarrow \boxed{(1-\alpha)} = \boxed{\frac{W.L}{p.Y}}$$

USING (2)

$$MPK = A\,\alpha \left(\frac{L}{K}\right)^{1-\alpha}$$

$$= A\,\alpha\, L^{1-\alpha}\, K^{-1+\alpha}$$

MULTIPLY & DIVIDE RIGHT HAND SIDE BY $K$

$$= \frac{\alpha}{K} \underbrace{A\, L^{1-\alpha}\, K^{\alpha}}_{Y}$$

$$\Longrightarrow MPK = \alpha \frac{Y}{K}$$

MULTIPLY BOTH SIDES BY $p$:

$$\underbrace{p.MPK}_{= r \text{ BY } (*)} = \alpha \frac{p.Y}{K}$$

$$\Longrightarrow \boxed{\alpha = \frac{r.K}{p.Y}}$$

# RETURNS TO SCALE

$$Y = F(K,L) = A \cdot K^{\alpha} \cdot L^{1-\alpha}$$

SINCE $\alpha + (1-\alpha) = 1$

$\Longrightarrow$ C.R.S. (CONSTANT RETURNS TO SCALE)

EX: ASSUME K & L
↑ BY 30%

$K_1 = 1.3\,K_0$
$L_1 = 1.3\,L_0$

WE WILL SHOW THAT:

$$Y_1 = 1.3\,Y_0$$

$$\boxed{Y_1} = F(K_1, L_1) =$$

$$= A\,(1.3\,K_0)^{\alpha} \cdot (1.3\,L_0)^{1-\alpha}$$

$$= (1.3)^{\alpha + 1-\alpha}\, \underbrace{A\,K_0^{\alpha}\,L_0^{1-\alpha}}_{Y_0}$$

$$= \boxed{1.3\,Y_0}$$

## REMARKS ABOUT RETURNS TO SCALE

① CRS $\Rightarrow$ AC CONSTANT (INDEPENDENT OF OUTPUT LEVEL)
$\Rightarrow$ THE "OPTIMAL" FIRM SIZE IS UNDETERMINED

② EASY TO WORK WITH SINCE WE CAN ASSUME THERE IS A SINGLE FIRM OR AS MANY FIRMS AS WORKERS.

---

## REMARK ABOUT COMPETITIVE EQUILIBRIUM AND PLANNING PROBLEM

THEY COINCIDE IN CASES LIKE THE ONES WE WILL BE STUDYING IN THE EARLY PART OF COURSE (NO EXTERNALITIES)

# CHANGE OF VARIABLES

$Y = F(k, L) = A\,k^{\alpha} L^{1-\alpha}$

ORIGINAL SYSTEM / PROD. FUNCTION
HAS 3 VARIABLES
DIFFICULT TO GRAPH & ANALYZE

**TASK**: REDUCE # VARIABLES
WE GET A "MODIFIED" SYSTEM IN ONLY 2 VARIABLES.

**DEEPER EXPLANATION**:
~~[illegible]~~ FIND A MODIFIED SYSTEM THROUGH THE CHANGE OF VARIABLES SUCH THAT IT HAS A STEADY STATE

EXAMPLE:
MODIFIED SYSTEM IS IN "PER WORKER / PER CAPITA" VARIABLES.

ORIGINAL:

$$Y = A \cdot K^{\alpha} \cdot L^{1-\alpha}$$

DIVIDE BOTH SIDES BY $L$

$$\frac{Y}{L} = A \, K^{\alpha} \, \frac{L^{\cancel{1}-\alpha}}{\cancel{L}}$$

$$\Rightarrow \frac{Y}{L} = A \, K^{\alpha} L^{-\alpha}$$

$$= A \left(\frac{K}{L}\right)^{\alpha}$$

i.e.

$$\underbrace{\frac{Y}{L}}_{y} = A \underbrace{\left(\frac{K}{L}\right)}_{k}^{\alpha}$$

THAT IS $y = \frac{Y}{L}$ = OUTPUT PER WORKER

$k = \frac{K}{L}$ = CAPITAL PER WORKER

$$\boxed{y = \underbrace{A\, h^{\alpha}}_{f(h)}}$$

REMEMBER

$$Y = F(K, L) = A\, K^{\alpha} L^{1-\alpha}$$

ORIGINAL PROD. FUNCTION

MODIFIED :

$$y = f(h) = A \cdot h^{\alpha}$$

WHERE $0 < \alpha < 1$

$$\frac{dy}{dh} = f'(h) = A\, \alpha\, h^{\alpha - 1} > 0$$

$$\frac{d^2 y}{dh^2} = A\, \alpha\, \underbrace{(\alpha - 1)}_{<0}\, h^{\alpha - 2} < 0$$